DEPARTMENT OF STATISTICS
AHMADU BELLO UNIVERSITY, ZARIA
2nd International Conference on Statistical Science and Data Analytics (ICSSDA 2026) Proceedings Paper

A Flexible Three Parameter Extended Sine Weibull Distribution: Mathematical Properties and Application to Rainfall Data

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Abstract

This paper introduces the Exponentiated Sine-Weibull (ESW) distribution, a flexible three-parameter extension of the classical Weibull distribution obtained by exponentiating the cumulative distribution function of the Sine-Weibull model. Several important mathematical properties of the proposed distribution are derived, including the probability density function, cumulative distribution function, survival function, hazard function, reverse hazard function, cumulative hazard function, quantile function, moments, moment generating function, probability weighted moments, and Shannon entropy. Model parameters are estimated using Maximum Likelihood Estimation (MLE) and Maximum Product of Spacings (MPS). A Monte Carlo simulation study demonstrates the consistency and reliability of both estimation methods, with bias and root mean square error decreasing as the sample size increases. The practical usefulness of the ESW distribution is illustrated using rainfall data and compared with the Weibull, Generalized Weibull, Sine-Weibull and Topp-Leone Modified Weibull distributions. Goodness-of-fit measures based on the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (KS), and Anderson-Darling (AD) statistics indicate that the proposed ESW distribution provides the best fit among the competing models. These findings demonstrate that the ESW distribution is a flexible and effective model for analyzing rainfall data and has potential applications to other environmental, reliability and lifetime datasets.

Keywords: Exponentiated G Family, Sine Weibull distribution, Maximum Product of Spacing, Maximum Likelihood Estimate.

1. Introduction

The development of flexible probability distributions has attracted considerable attention in statistical research because classical distributions often fail to adequately capture complex data characteristics such as heavy tails, high skewness, bimodality, and diverse hazard rate shapes. Although the classical Weibull distribution has been widely used in reliability, survival analysis, engineering, and other applied fields due to its mathematical simplicity and flexibility, its two-parameter structure may be insufficient for modelling datasets with more intricate distributional patterns. To address these limitations, this study proposes the Exponentiated Sine-Weibull (ESW) distribution, a flexible three-parameter extension of the classical Weibull distribution obtained by exponentiating the cumulative distribution function of the Sine-Weibull distribution. The additional exponentiation parameter enhances the flexibility of the model while preserving important statistical properties such as the probability density function, cumulative distribution function, and hazard rate.

The Sine-Weibull distribution proposed by Faruk *et al*. (2024) introduced a flexible lifetime distribution by applying a sine transformation to the Weibull cumulative distribution function. Subsequently, Isa *et al*. (2023) proposed the Sine Type II Topp-Leone-G family, providing a general framework for constructing new distributions from arbitrary baseline models. This framework was further extended by Isa *et al*. (2024a) through the Sine-Topp-Leone Exponentiated-G family, which incorporated an additional exponentiation parameter to increase flexibility in modelling density and hazard functions. The methodology was later extended to survival regression through a Weibull accelerated failure time model (Isa *et al*., 2025), demonstrating its applicability in modelling lifetime data. Similar sine-based constructions have also been successfully applied to other baseline distributions, including the Burr XII distribution (Isa *et al*., 2024b) and the Topp-Leone Exponentiated distribution (Sule *et al*., 2023). These developments provide the theoretical and methodological foundation for the proposed Exponentiated Sine-Weibull distribution.

Parallel to the development of sine-based families, numerous extensions of the Weibull distribution have been proposed to improve its flexibility. The exponentiated Weibull distribution introduced by Pal *et al*. (2006) laid the foundation for a wide range of generalized Weibull models with additional parameters and enhanced distributional characteristics. Subsequently, Cordeiro *et al*. (2013) proposed the Beta Exponentiated Weibull distribution, while Pinho *et al*. (2012) developed the Gamma Exponentiated Weibull distribution, both of which provided greater flexibility in modelling density and hazard functions. Further extensions include the Exponentiated Inverted Weibull distribution (Flaih *et al*., 2012), the Exponentiated Generalized Inverse Weibull distribution (Elbatal & Muhammed, 2014), the Exponentiated Generalized Weibull distribution (Oguntunde *et al*., 2015), and the Exponentiated Modified Weibull distribution (Sarhan & Apaloo, 2013). These studies demonstrate the continuing effort to develop more flexible Weibull-based models capable of accommodating a broad range of distributional behaviours. Motivated by these developments, this paper introduces the Exponentiated Sine-Weibull distribution, derives its mathematical properties, investigates parameter estimation using Maximum Likelihood Estimation and Maximum Product of Spacings, evaluates the performance of the estimators through Monte Carlo simulation, and illustrates the usefulness of the proposed model using a real dataset.

2. Methodology

2.1. Sine-Weibull Distribution

The CDF of the Sine-Weibull distribution as proposed by Faruk *et al*. (2024) is given by:

$$H(x;\beta,\theta)=\sin\left\{\frac{\pi}{2}\left[1-e^{-\left(\frac{x}{\theta}\right)^{\beta}}\right]\right\}, x,\beta,\theta>0 \tag{1}$$

And the corresponding PDF is obtained by differentiating the CDF in equation (1) as follows:

$$h(x;\beta,\theta)=\frac{\pi}{2}.\frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-(x/\theta)^{\beta}}\cos\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right) \quad (2),$$

where $\boldsymbol{x>0}$, $\boldsymbol{\beta>0}$ is a shape parameter and $\boldsymbol{\theta>0}$ is a scale parameter.

2.2. The Proposed Exponentiated Sine-Weibull Distribution

The proposed Exponentiated Sine-Weibull distribution is obtained by raising the CDF of the Sine-Weibull distribution to a new positive power $\alpha > 0$ (the exponentiation parameter). The CDF of the proposed model is given by:

$$F(x;\alpha,\beta,\theta) = [H(x;\beta,\theta)]^{\alpha} = \left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\right]^{\alpha}, \quad x,\alpha,\beta,\theta > 0 \tag{3}$$

and the PDF is obtained as follows:

$$f(x;\alpha,\beta,\theta) = \frac{d}{dx}F(x;\alpha,\beta,\theta) = \alpha\, h(x;\beta,\theta)[H(x;\beta,\theta)]^{\alpha-1} \tag{5}$$

Substituting $H(x;\beta,\theta)$ and $h(x;\beta,\theta)$ into equation (3)

$$f(x;\alpha,\beta,\theta) = \frac{\pi}{2}\frac{\alpha\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^{\beta}} \cos\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\right]^{\alpha-1} \tag{6}$$

This is the closed-form PDF of the Exponentiated Sine-Weibull distribution (three-parameter model).

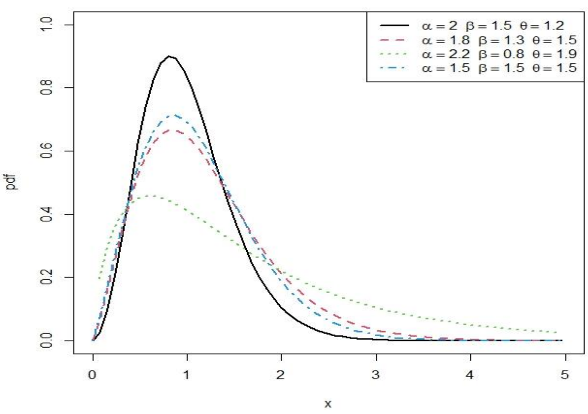

Figure 1. PDF plot of ESW Distribution

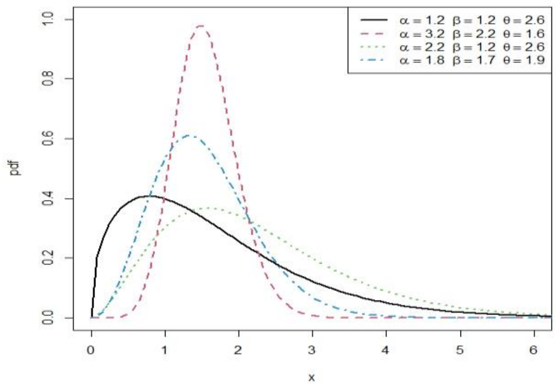

Figure 2. Hazard plot of ESW Distribution

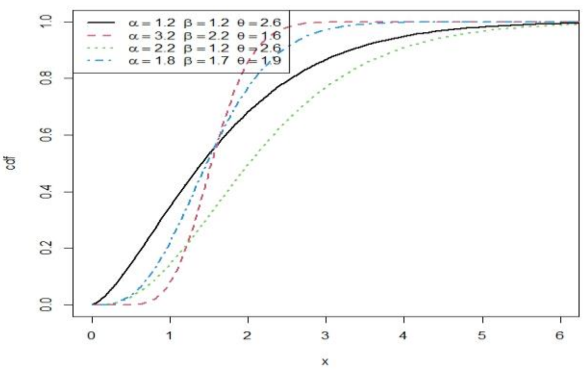

Figure 3. CDF plot of ESW Distribution

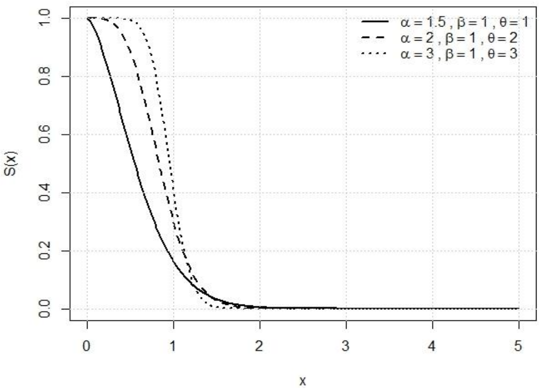

Figure 3. Survival plot of ESW Distribution

Figure 1 shows that the PDF of the ESW distribution is sharply peaked and symmetric, indicating that the proposed model can capture datasets with high kurtosis and also skewed data.

2.2.1. Hazard Function

The hazard function h(x) measures the instantaneous rate of occurrence of an event at time or value x, given that the event has not occurred before x (Sadiq *et al*. (2022); Sadiq *et al*. (2023a); Sadiq *et al*. (2023b); Sadiq *et al*. (2022c); Sadiq *et al*. (2024); Sadiq *et al*. (2025)). In the context of the Exponentiated Sine Weibull distribution, the hazard rate is expressed as:

2.2.2. Hazard Function

$$h(x)=\frac{\alpha\left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\right]^{\alpha-1}\cdot\frac{\pi}{2}\cdot\frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-(x/\theta)^{\beta}}\cdot\cos\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)}{1-\left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\right]^{\alpha}} \tag{7}$$

2.2.3. Reverse Hazard Function

The reverse hazard function, denoted as r(x), measures the instantaneous rate of failure at value x, given that the failure has already occurred by x. For the Exponentiated Sine Weibull distribution, the reverse hazard function is given by:

$$r(x)=\frac{\theta f(x)F(x)^{\theta-1}}{F(x)^{\theta}}$$

where f(x) and F(x) are the probability density function and cumulative distribution function of the baseline Sine-Weibull distribution, respectively.

$$r(x)=\frac{\alpha\left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\right]^{\alpha-1}\cdot\frac{\pi}{2}\cdot\frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-(x/\theta)^{\beta}}\cdot\cos\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)}{\left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\right]^{\alpha}} \tag{8}$$

2.2.4. Cumulative Hazard Function

The cumulative hazard function H(x) represents the accumulated risk of an event up to a specific value x **(**Sadiq *et al*. (2022); Sadiq *et al*. (2023a); Sadiq *et al*. (2023b); Sadiq *et al*. (2022c); Sadiq *et al*. (2024); Sadiq *et al*. (2025)**)**. For the Exponentiated Sine Weibull distribution, it is defined as:

$$H_{\text{cum}}(x)=-\ln S(x)=-\ln\left(1-\left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\right]^{\alpha}\right) \tag{9}$$

2.2.5. Quantile Function

Let $p\in(0,1)$ be a given cumulative probability. The quantile x_p satisfies $F(x_p)=p$.

$$F(x)=\left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\right]^{\alpha}$$

Set $F(x_p)=p$:

$$\left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x/\theta)^{\beta}}\right]\right)\right]^{\alpha}=p$$

Take the α-th root:

$$\left[\sin\left(\frac{\pi}{2}\left[1-e^{-(x_p/\theta)^{\beta}}\right]\right)\right]=p^{1/\alpha}$$

Apply the inverse sine function because the sine term is positive for $x > 0$, so arcsin is valid):

$$\frac{\pi}{2}\left[1-e^{-(x_p/\theta)^\beta}\right]=\arcsin\left(p^{1/\alpha}\right)$$

Solve for the exponential term:

$$1-e^{-(x_p/\theta)^\beta}=\frac{2}{\pi}\arcsin\left(p^{1/\alpha}\right)$$

$$e^{-(x_p/\theta)^\beta}=1-\frac{2}{\pi}\arcsin\left(p^{1/\alpha}\right)$$

Take the natural logarithm:

$$-\left(\frac{x_p}{\theta}\right)^\beta=\ln\left(1-\frac{2}{\pi}\arcsin\left(p^{1/\alpha}\right)\right).$$

Multiply by -1:

$$\left(\frac{x_p}{\theta}\right)^\beta=-\ln\left(1-\frac{2}{\pi}\arcsin\left(p^{1/\alpha}\right)\right).$$

Raise both sides to the power $1/\beta$ and multiply by θ:

$$x_p=\theta\left[-\ln\left(1-\frac{2}{\pi}\arcsin\left(p^{1/\alpha}\right)\right)\right]^{1/\beta},\quad 0<p<1 \tag{10}$$

Equation (10) is the quantile function of the Exponentiated Sine-Weibull distribution.

2.2.6. Mixture Representation

By applying power series expansion, the PDF of the proposed model can be written as

$$f(x;\alpha,\beta,\theta)=\sum_{i,j,k=0}^{\infty}c_i\frac{(-1)^{i+j+k}}{(2j)!}\left(\frac{\alpha\beta}{\theta^\beta}\right)\left(\frac{\pi}{2}\right)^{2j+\alpha}x^{\beta-1}e^{-(k+1)\left(\frac{x}{\theta}\right)^\beta}$$

where the coefficients c_n are determined by the expansion of

$$c_n=\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m+1}\right)^{\alpha-1}$$

The first few coefficients are:

$$c_0=1,\quad c_1=\frac{1-a}{6},\quad c_2=\frac{(a-1)(a-2)}{120}+\frac{(a-1)^2}{72},\quad \ldots$$

The PDF can be reduced to

$$f(x;\alpha,\beta,\theta)=\sum_{i,j,k=0}^{\infty}\Phi_{i,j,k}x^{\beta-1}e^{-(k+1)\left(\frac{x}{\theta}\right)^\beta}$$

Where

$$\Phi_{i,j,k} = c_i \frac{(-1)^{i+j+k}}{(2j)!} \left(\frac{\alpha\beta}{\theta^\beta}\right)\left(\frac{\pi}{2}\right)^{2j+\alpha}$$

For integer α, the series terminates and reduces to a finite trigonometric polynomial. For non-integer α, the series is infinite and converges for $|\ x\ |< \pi$.

2.3. Mathematical Properties

2.3.1. The r-th Moment

The r-th moment is $\mu_r' = \mathbb{E}[X^r] = \int_0^\infty x^r\, f(x)\, dx$.
Substitute the series:

$$\mu_r' = \sum_{i,j,k=0}^{\infty} \Phi_{i,j,k} \int_0^\infty x^{\beta-1+r}\ e^{-(k+1)(\frac{x}{\theta})^\beta}\ dx.$$

Use the substitution $u = (\frac{x}{\theta})^\beta$, i.e. $x = \theta u^{\frac{1}{\beta}}$, $dx = \frac{\theta}{\beta} u^{1/\beta-1} du$. Then

$$x^{\beta-1+r} dx = \theta^{\beta+r} u^{r/\beta} \frac{du}{\beta}.$$

The integral becomes

$$\int_0^\infty x^{\beta-1+r}\ e^{-(k+1)(x/\theta)^\beta} dx = \frac{\theta^{\beta+r}}{\beta} \int_0^\infty u^{r/\beta}\ e^{-(k+1)u} du.$$

The integral is the Gamma function:

$$\int_0^\infty u^{r/\beta}\ e^{-(k+1)u} du = \frac{\Gamma\left(1+\frac{r}{\beta}\right)}{(k+1)^{1+r/\beta}}.$$

Hence

$$\mu_r' = \sum_{i,j,k=0}^{\infty} \Phi_{i,j,k}\ \frac{\theta^{\beta+r}}{\beta}\ \frac{\Gamma\left(1+\frac{r}{\beta}\right)}{(k+1)^{1+r/\beta}}.$$

$$\mu_r' = \theta^r\, \Gamma\left(1+\frac{r}{\beta}\right) \sum_{i,j,k=0}^{\infty} \Phi_{i,j,k}\ \frac{\theta^\beta}{\beta}\ (k+1)^{-(1+r/\beta)}$$

2.3.2. Moment Generating Function (MGF)

The MGF is $(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx}\, f(x)\, dx$. Expand the exponential in a power series:

$$e^{tx} = \sum_{m=0}^{\infty} \frac{t^m}{m!} x^m.$$

Then

$$M(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \mathbb{E}[X^m] = \sum_{m=0}^{\infty} \frac{t^m}{m!} \mu_m'.$$

Using the expression for μ_m' from above,

$$M(t)=\sum_{m=0}^{\infty}\frac{t^m}{m!}\;\theta^m\,\Gamma\left(1+\frac{m}{\beta}\right)\sum_{i,j,k}\Phi_{i,j,k}\frac{\theta^{\beta}}{\beta}(k+1)^{-\left(1+\frac{m}{\beta}\right)}$$

This series representation is valid for all t where the sum converges.

2.3.3. Probability Weighted Moments (PWM)

A common definition of probability weighted moments is

$$\omega_{r,s}=\mathbb{E}[X^r(F(X))^s]=\int_0^{\infty}x^r\,(F(x))^s f(x)\,dx,$$

where $F(x)$ is the cumulative distribution function. Integrate the PDF term by term:

$$F(x)=\int_0^x f(t)\,dt=\sum_{i,j,k}\Phi_{i,j,k}\int_0^x t^{\beta-1}\,e^{-(k+1)(t/\theta)^{\beta}}dt.$$

Using the same substitution $u=(t/\theta)^{\beta}$,

$$\int_0^x t^{\beta-1}\,e^{-(k+1)(t/\theta)^{\beta}}dt=\frac{\theta^{\beta}}{\beta}\int_0^{(x/\theta)^{\beta}}e^{-(k+1)u}\,du=\frac{\theta^{\beta}}{\beta}\cdot\frac{1-e^{-(k+1)(x/\theta)^{\beta}}}{k+1}.$$

Thus

$$F(x)=\sum_{i,j,k}\Phi_{i,j,k}\frac{\theta^{\beta}}{\beta(k+1)}\left[1-e^{-(k+1)(x/\theta)^{\beta}}\right].$$

$$(F(x))^s=\sum_{\{m_k\}}\frac{s!}{\prod_k m_k\,!}\prod_k\left(\sum_{i,j:\,k\text{ fixed}}\Phi_{i,j,k}\frac{\theta^{\beta}}{\beta(k+1)}\right)^{m_k}\left(1-e^{-(k+1)(x/\theta)^{\beta}}\right)^{m_k},$$

where the outer sum runs over all non-negative integer vectors $(m_0,m_1,m_2,\dots)$ with $\sum_k m_k=s$. The inner sum over i,j for a fixed k is the total coefficient for that k; denote

$$\Psi_k=\sum_{i,j}\Phi_{i,j,k}\frac{\theta^{\beta}}{\beta(k+1)}.$$

Then

$$F(x)=\sum_{k=0}^{\infty}\Psi_k\left(1-e^{-(k+1)(x/\theta)^{\beta}}\right),$$

and

$$(F(x))^s=\sum_{\substack{m_0,m_1,\dots\ge 0\\ \sum m_k=s}}\frac{s!}{\prod_k m_k\,!}\prod_k\Psi_k^{m_k}\left(1-e^{-(k+1)(x/\theta)^{\beta}}\right)^{m_k}.$$

Now multiply by $x^r f(x)$ and integrate. Since $f(x)$ also contains a series, the overall expression becomes a sum over indices. The resulting integral is of the form.

$$I=\int_0^{\infty}x^{\beta-1+r}\,e^{-(k_0+1)(x/\theta)^{\beta}}\prod_k\left(1-e^{-(k+1)(x/\theta)^{\beta}}\right)^{m_k}dx.$$

Each factor $\left(1-e^{-(k+1)u}\right)^{m_k}$ can be expanded using the binomial theorem:

$$\left(1-e^{-(k+1)u}\right)^{m_k}=\sum_{p_k=0}^{m_k}\binom{m_k}{p_k}(-1)^{p_k}e^{-p_k(k+1)u}.$$

Thus, the product becomes a sum of exponentials e^{-Qu} where $Q = (k_0 + 1) + \sum_k p_k\,(k + 1)$. The integral then reduces to a Gamma function:

$$\int_0^\infty u^{r/\beta}\, e^{-Qu} du = \frac{\Gamma(1 + r/\beta)}{Q^{1+r/\beta}}.$$

Consequently, the probability-weighted moment can be expressed as

$$\omega_{(r,s)} = \sum_{k_0=0}^{\infty} \Phi_{(i_0,j_0,k_0)} \frac{\theta^{\beta+r}}{\beta} \sum_{\substack{m_0,m_1,\ldots\geq 0 \\ \sum m_k = s}} \frac{s!}{\prod_k m_k!} \prod_k \Psi_k^{m_k} \sum_{p_0,p_1,\ldots} \prod_k \binom{m_k}{p_k} (-1)^{\sum p_k} \frac{\Gamma\left(1+\frac{r}{\beta}\right)}{\left(k_0+1+\sum_k p_k(k+1)\right)^{1+r/\beta}}$$

This is a general expression. For practical use, the inner sums truncate when s is small.

2.3.4. Shannon Entropy

Shannon entropy is defined as $= -\int_0^\infty f(x) \ln f(x)\, dx$. Using the series representation, write $f(x) = \sum_n A_n\, x^{\beta-1} e^{-b_n (x/\theta)^\beta}$ where we combine indices into a single index n corresponding to the ple (i, j, k) with $b_n = k + 1$ and $A_n = \Phi_{i,j,k}$. Then

$$\ln f(x) = \ln\left(\sum_n A_n\, x^{\beta-1} e^{-b_n (x/\theta)^\beta}\right).$$

Direct integration of $f \ln f$ is difficult due to the logarithm of a sum. A common approach is to use the series expansion of the logarithm:

$$\ln f(x) = \ln(x^{\beta-1}) + \ln\left(\sum_n A_n\, e^{-b_n (x/\theta)^\beta}\right).$$

Alternatively, we can express entropy as

$$H = -\sum_n A_n \int_0^\infty x^{\beta-1}\, e^{-b_n (x/\theta)^\beta} \ln f(x)\, dx.$$

2.4. Parameter Estimation

The parameter estimation of the model was done using maximum likelihood (MLE) and maximum product of spacings (MPS) methods.

2.4.1. Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be an independent random sample from the Exponentiated Sine-Weibull distribution with parameters $\alpha > 0$, $\beta > 0$, $\theta > 0$. The log-likelihood function is given by:

$$l = n\log\left(\frac{\pi}{2}\right) + n\log\alpha + n\log\beta - n\log\theta + (\beta-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\theta}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^\beta + \sum_{i=1}^{n}\log\cos\left(\frac{\pi}{2}\left[1 - e^{-(x_i/\theta)^\beta}\right]\right)$$

$$+(\alpha-1)\sum_{i=1}^{n}\log\sin\left(\frac{\pi}{2}\left[1 - e^{-(x_i/\theta)^\beta}\right]\right) \tag{11}$$

Differentiating equation (11) with respect to α gives the following:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\sin\left(\frac{\pi}{2}\left[1 - e^{-(x_i/\theta)^\beta}\right]\right) \tag{12}$$

Differentiating equation (11) with respect to β gives the following:

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}\log\left(\frac{x_i}{\theta}\right) + \frac{\pi}{2}\sum_{i=1}^{n} e^{-(x_i/\theta)^{\beta}}\left(\frac{x_i}{\theta}\right)^{\beta}\log\left(\frac{x_i}{\theta}\right)$$

$$\times\left\{(\alpha-1)\cot\left(\frac{\pi}{2}\left[1-e^{-(x_i/\theta)^{\beta}}\right]\right) - \tan\left(\frac{\pi}{2}\left[1-e^{-(x_i/\theta)^{\beta}}\right]\right)\right\} \tag{13}$$

Differentiating equation (11) with respect to θ gives the following:

$$\frac{\partial l}{\partial \theta} = \frac{n\beta}{\theta} + \frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} + \frac{\pi}{2}\frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}$$

$$\times e^{-u_i}\left[\tan\left(\frac{\pi}{2}\left[1-e^{-(x_i/\theta)^{\beta}}\right]\right) - (\alpha-1)\cot\left(\frac{\pi}{2}\left[1-e^{-(x_i/\theta)^{\beta}}\right]\right)\right] \tag{14}$$

Equations (12), (13) and (14) give the maximum likelihood estimators of the parameters α, β and θ.

3. Results and Discussions

3.1. Consistency of the Parameter Estimates using MLE

A Monte Carlo simulation study was conducted to evaluate the performance of the maximum likelihood estimators of the Exponentiated Sine Weibull (ESW) distribution. Two sets of true parameter values, (α = 2.0, β = 1.5, θ = 1.0), (α = 3, β = 2, θ = 1.5), (α = 2.2, β = 1.2, θ =0.8) and (α = 2.5, β = 2.0, θ = 1.2) were considered. For each case, 10,000 replications were generated at sample sizes n = 100, 200, 500, and 1000. The results show that as sample size increases, both bias and RMSE decrease substantially, confirming that the estimators are consistent and reliable.

Table 1. Simulation Results for Consistency of Parameter Estimates using MLE

n	Properties	$\alpha = 2.0$	$\beta = 1.5$	$\theta = 1.0$	$\alpha = 3.0$	$\beta = 2.0$	$\theta = 1.5$
100	Estimate	2.2125	1.6400	0.9987	3.5374	2.1765	1.4944
	Bias	0.2125	0.1400	-0.0013	0.5374	0.1765	-0.0056
	RMSE	1.4957	0.1911	0.0293	5.9421	0.3408	0.0475
200	Estimate	2.3797	1.5548	0.9831	3.9381	2.0649	1.4744
	Bias	0.3797	0.0548	-0.0169	0.9381	0.0649	-0.0256
	RMSE	3.4183	0.0956	0.0242	19.1604	0.1722	0.0416
500	Estimate	2.1217	1.4990	0.9835	3.2511	1.9969	1.4781
	Bias	0.1217	-0.0010	-0.0165	0.2511	-0.0031	-0.0219
	RMSE	0.3119	0.0377	0.0091	1.0196	0.0686	0.0146
1000	Estimate	2.1205	1.4781	0.9858	3.2340	1.9662	1.4792
	Bias	0.1205	-0.0219	-0.0142	0.2340	-0.0338	-0.0208
	RMSE	0.1460	0.0154	0.0042	0.4679	0.0283	0.0067

Table 2. Simulation Results for Consistency of Parameter Estimates using MLE

N	Properties	$\alpha = 2.2$	$\beta = 1.2$	$\theta = 0.8$	$\alpha = 2.5$	$\beta = 1.0$	$\theta = 1.2$
100	Estimate	2.3590	1.2958	0.8082	2.7257	1.0773	1.2211

	Bias	0.1590	0.0958	0.0082	0.2257	0.0773	0.0211
	RMSE	1.6144	0.1013	0.0258	2.4615	0.0697	0.0865
200	Estimate	2.3920	1.2377	0.7899	2.7411	1.0310	1.1863
	Bias	0.1920	0.0377	-0.0101	0.2411	0.0310	-0.0137
	RMSE	1.0319	0.0387	0.0135	1.5434	0.0265	0.0442
500	Estimate	2.3677	1.1874	0.7810	2.7081	0.9892	1.1668
	Bias	0.1677	-0.0126	-0.0190	0.2081	-0.0108	-0.0332
	RMSE	0.3925	0.0208	0.0089	0.5694	0.0145	0.0301
700	Estimate	2.3468	1.1851	0.7841	2.6829	0.9868	1.1706
	Bias	0.1468	-0.0149	-0.0159	0.1829	-0.0132	-0.0294
	RMSE	0.2544	0.0137	0.0050	0.3723	0.0097	0.0172
1000	Estimate	2.2518	1.2014	0.7986	2.5654	1.0008	1.1974
	Bias	0.0518	0.0014	-0.0014	0.0654	0.0008	-0.0026
	RMSE	0.1119	0.0084	0.0030	0.1610	0.0060	0.0106

We assessed the performance of the MLE method in recovering the true parameters of the Exponentiated Sine Weibull distribution using a Monte Carlo simulation approach. Specifically, 10,000 random samples were generated for each case across different sample sizes (n = 100, 200, 500, and 1000) and four distinct sets of true parameter values. Parameter estimates were obtained for each sample, and the average bias and Root Mean Square Error (RMSE) were computed to evaluate estimation accuracy and precision. The results show that both bias and RMSE decrease as the sample size increases. At smaller sample sizes, such as n = 100, the estimates are more variable and deviate more from the true values, whereas at larger sample sizes, such as n = 1000, the estimates are closer to the true parameters and more stable. This pattern demonstrates that the MLE estimators are consistent, as their accuracy and reliability improve with increasing sample size.

3.2. Maximum Product of Spacing (MPS) Estimation

The Maximum Product of Spacings method is an alternative to MLE that maximises the product of differences between consecutive CDF values. It is particularly useful when the likelihood is unbounded or when dealing with heavy-tailed distributions.

Let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ be the ordered sample from the Exponentiated Sine-Weibull distribution with CDF $F(x;\alpha,\beta,\theta)$. Define the spacings as:

$$D_i(\alpha,\beta,\theta) = F(x_{(i)};\alpha,\beta,\theta) - F(x_{(i-1)};\alpha,\beta,\theta), i = 1,2,\ldots,n+1$$

where $F(x_{(0)}) = 0$ and $F(x_{(n+1)}) = 1$. The product of spacings is $\prod_{i=1}^{n+1} D_i$. The MPS estimators are obtained by maximising the geometric mean of the spacings, or equivalently, by minimising the negative log-spacings function:

$$\ell_{\mathrm{MPS}}(\alpha,\beta,\theta) = -\sum_{i=1}^{n+1} \ln D_i(\alpha,\beta,\theta) \tag{15}$$

For the Exponentiated Sine-Weibull distribution, the CDF is:

$$F(x;\alpha,\beta,\theta)=\left[\sin\left(\frac{\pi}{2}\left(1-e^{-(x/\theta)^{\beta}}\right)\right)\right]^{\alpha}$$

Thus, the spacings become:

$$D_i=\left[\sin\left(\frac{\pi}{2}\left(1-e^{-(x_{(i)}/\theta)^{\beta}}\right)\right)\right]^{\alpha}-\left[\sin\left(\frac{\pi}{2}\left(1-e^{-(x_{(i-1)}/\theta)^{\beta}}\right)\right)\right]^{\alpha}$$

with $x_{(0)}=0$ and $x_{(n+1)}=\infty$ (so $F(x_{(n+1)})=1$).

The MPS estimates $\hat{\alpha}_{\mathrm{MPS}},\hat{\beta}_{\mathrm{MPS}},\hat{\theta}_{\mathrm{MPS}}$ are the values that minimise (15). Taking partial derivatives of ℓ_{MPS} with respect to each parameter yields the following estimating equations:

Differentiating with respect to α

$$\frac{\partial\ell_{\mathrm{MPS}}}{\partial\alpha}=-\sum_{i=1}^{n+1}\frac{1}{D_i}\frac{\partial D_i}{\partial\alpha}=0 \tag{16}$$

where

$$\frac{\partial D_i}{\partial\alpha}=D_i\cdot\ln\left[\sin\left(\frac{\pi}{2}\left(1-e^{-(x_{(i)}/\theta)^{\beta}}\right)\right)\right]-D_{i-1}\cdot\ln\left[\sin\left(\frac{\pi}{2}\left(1-e^{-(x_{(i-1)}/\theta)^{\beta}}\right)\right)\right]$$

Differentiating with respect to β:

$$\frac{\partial\ell_{\mathrm{MPS}}}{\partial\beta}=-\sum_{i=1}^{n+1}\frac{1}{D_i}\frac{\partial D_i}{\partial\beta}=0 \tag{17}$$

Differentiating with respect to θ:

$$\frac{\partial\ell_{\mathrm{MPS}}}{\partial\theta}=-\sum_{i=1}^{n+1}\frac{1}{D_i}\frac{\partial D_i}{\partial\theta}=0 \tag{18}$$

Equations (16), (17) and (18) give the MPS estimators of the parameters α,β,θ. These equations are typically solved numerically using optimization algorithms such as Newton-Raphson or L-BFGS-B.

3.3. Consistency of the Parameter Estimate using MPS

A Monte Carlo simulation study was conducted to evaluate the performance of the Maximum Product of Spacings (MPS) estimators for the Exponentiated Sine Weibull (ESW) distribution. Four sets of true parameter values, (α = 2.0, β = 1.5, θ = 1.0), (α = 3.0, β = 2.0, θ = 1.5), (α = 2.2, β = 1.2, θ = 0.8), and (α = 2.5, β = 2.0, θ = 1.2), were considered. For each configuration, 10,000 replications were generated at sample sizes n = 100, 200, 500, and 1000. Parameter estimates were obtained using the MPS method, and their performance was assessed using bias and Root Mean Square Error (RMSE). The results indicate that as the sample size increases, both bias and RMSE decrease markedly, demonstrating improved estimation accuracy and stability. This behaviour confirms that the MPS estimators are consistent and reliable for estimating the parameters of the ESW distribution.

Table 3. Simulation Results for Consistency of Parameter Estimates using MPS

n	**Properties**	$\alpha=2.0$	$\beta=1.5$	$\theta=1.0$	$\alpha=3.0$	$\beta=2.0$	$\theta=1.5$
100	Estimate	2.09500	1.56100	1.01550	3.29610	2.08000	1.51010
	Bias	0.09500	0.06100	0.01550	0.29610	0.08000	0.01010
	RMSE	1.04700	0.36900	0.16190	2.11210	0.50000	0.20640
200	Estimate	2.14230	1.51570	0.99440	3.27100	2.02830	1.49410
	Bias	0.14230	0.01570	-0.00560	0.27100	0.02830	-0.00590

	RMSE	0.85660	0.23500	0.11610	1.63300	0.31370	0.14370
500	Estimate	2.13510	1.46880	0.98300	3.24940	1.96190	1.47770
	Bias	0.13510	-0.03120	-0.01700	0.24940	-0.03810	-0.02230
	RMSE	0.52820	0.17610	0.09120	0.94410	0.23690	0.11450
700	Estimate	2.12700	1.46880	0.98530	3.23820	1.95810	1.47900
	Bias	0.12700	-0.03120	-0.01470	0.23820	-0.04190	-0.02100
	RMSE	0.43840	0.14590	0.06910	0.79610	0.19950	0.08960
1000	Estimate	2.04340	1.49280	0.99970	3.07720	1.99200	1.49820
	Bias	0.04340	-0.00720	-0.00030	0.07720	-0.00800	-0.00180
	RMSE	0.29160	0.11350	0.05350	0.51580	0.15670	0.06800

Table 4. Simulation Results for Consistency of Parameter Estimates using MPS

n	Properties	$\boldsymbol{\alpha = 2.2}$	$\boldsymbol{\beta = 1.2}$	$\boldsymbol{\theta = 0.8}$	$\boldsymbol{\alpha = 2.5}$	$\boldsymbol{\beta = 1.0}$	$\boldsymbol{\theta = 1.2}$
100	Estimate	2.3241	1.2485	0.8186	2.6774	1.0402	1.2394
	Bias	0.1241	0.0485	0.0186	0.1774	0.0402	0.0394
	RMSE	1.2217	0.2955	0.1641	1.5177	0.2473	0.3022
200	Estimate	2.3633	1.2136	0.7968	2.6993	1.0125	1.1991
	Bias	0.1633	0.0136	-0.0032	0.1993	0.0125	-0.0009
	RMSE	0.9901	0.1874	0.1161	1.2109	0.156	0.2112
500	Estimate	2.3549	1.1757	0.7838	2.6876	0.9803	1.1722
	Bias	0.1549	-0.0243	-0.0162	0.1876	-0.0197	-0.0278
	RMSE	0.6033	0.1409	0.0925	0.7236	0.1176	0.1702
700	Estimate	2.3465	1.1751	0.7852	2.6784	0.9792	1.1730
	Bias	0.1465	-0.0249	-0.0148	0.1784	-0.0208	-0.0270
	RMSE	0.5025	0.1171	0.0705	0.6059	0.0984	0.1308
1000	Estimate	2.2494	1.1945	0.7999	2.5592	0.9956	1.2001
	Bias	0.0494	-0.0055	-0.0001	0.0592	-0.0044	0.0001
	RMSE	0.3329	0.0914	0.0550	0.3983	0.0770	0.1031

Tables 3 and 4 indicate that the closer the MPS parameter estimates are to the true values, the larger the sample size is and the smaller the bias and the RMSE are. Smaller samples are more varied and erratic, whereas larger samples are more precise and less prone to error. This general tendency proves that the MPS estimators are consistent and reliable for estimating the parameters of the Exponentiated Sine Weibull distribution.

3.3. Application

In order to illustrate the practical usefulness of the proposed Exponentiated Sine Weibull (ESW) distribution, it was utilised on a real-life dataset comprising monthly rainfall data.

Table 5. Goodness-of-Fit Comparison of Competing Models for Rainfall Data

Models	α	β	θ	-LL	AIC	BIC	KS	p-value	AD
ESW	8.6845	0.28470	14.8451	42.4809	90.2657	90.5040	0.072	0.680	0.45
GW	0.3239	15.4784	4.15330	50.9857	251.282	249.442	0.215	0.009	3.87
SW	-	0.31440	14.3594	51.3250	105.970	106.129	0.198	0.020	2.94
TLMW	0.0104	0.01729	0.68767	50.3281	106.656	111.406	0.176	0.04	2.13

The ESW distribution provides the best fit of the rainfall data. It has the lowest AIC (90.2657) and BIC (90.5040), the smallest KS statistic (0.072), the lowest AD value (0.45) and the only non-significant p-value (0.680 > 0.05), indicating no evidence of lack of fit. In contrast, all four models, that is GW, SW and TLMW provide a p-value less than 0.05 which means that their fit is not statistically significant.

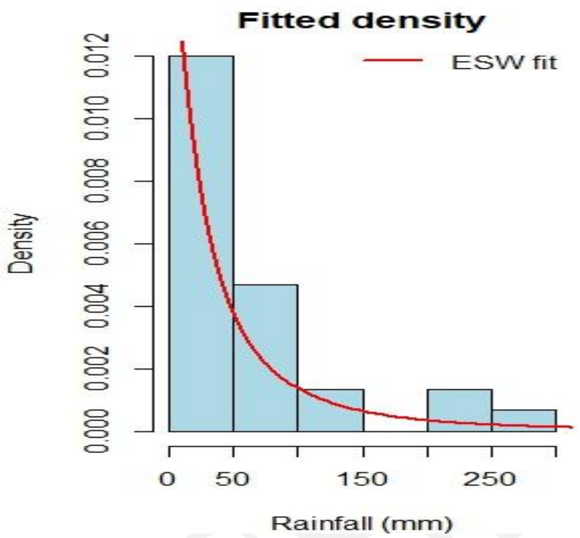

Figure 5. Fitted density plot

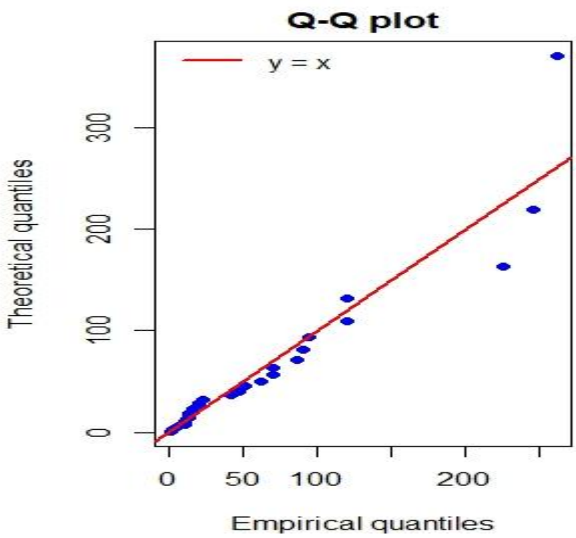

Figure 6. QQ-Plot

The fitted density plot reveals that the ESW distribution is able to capture the right-skewed distribution of the rainfall data, particularly in the low and moderate ranges, with a slight deviation in the extreme tail. The QQ plot shows that there is a good overall fit, as most of the points are located close to the reference line.

3. Conclusion

The proposed study presented the Exponentiated Sine Weibull (ESW) distribution as an extension of the classical Weibull model, which was designed to be flexible enough to effectively capture complex data characteristics such as skewness, heavy tails, and a variety of shapes of hazard rates. Theoretical properties of the model, such as the probability density function, cumulative distribution function, survival and hazard functions, quantile function, as well as the moments, were derived. Maximum Likelihood Estimation (MLE) and Maximum Product of Spacings (MPS) were used to estimate parameters, and a Monte Carlo simulation study confirmed the consistency and reliability of both methods, as bias and RMSE decreased with increasing sample size.

The suitability of the ESW distribution was illustrated with monthly rainfall statistics of Borno State in North-East Nigeria. The findings indicated that the ESW model gave the best fit among competing models based on goodness-of-fit measures like AIC, BIC, KS and AD statistics, with a non-significant p-value indicating that it had an adequate fit. By and large, the ESW distribution is a strong and effective model for environmental data analysis and can be applied to other fields.

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