DEPARTMENT OF STATISTICS
AHMADU BELLO UNIVERSITY, ZARIA
2nd International Conference on Statistical Sciences and Data Analytics (ICSSDA 2026) Proceedings Paper

On Some Properties of the Topp–Leone Exponential–Pareto Distribution with Applications to Real-Life Datasets

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Abstract

Modelling real-life datasets using probability distributions is of great importance across various fields, including science, medicine, engineering, and beyond. Numerous probability distributions have been employed to model data that exhibit skewness. In this study, we introduce a new skewed distribution—the Topp–Leone Exponential–Pareto (TLE–P) distribution, developed to better fit certain skewed datasets which existing distributions fail to adequately capture. The proposed distribution is constructed by extending the Pareto distribution via the Topp–Leone Exponential-*G* family of distributions. The cumulative distribution function, probability density function and several key mathematical properties, including moments, quantile function, and order statistics, are derived for the TLE–P distribution. For the simulation study, both the Maximum Likelihood Estimation (MLE) and Maximum Product of Spacing (MPS) methods are employed. The simulation results indicate that the parameter estimates of the TLE–P distribution are consistent, as the bias and root mean square error (RMSE) values approach zero. Additionally, two real datasets are analysed to assess the flexibility of the TLE–P distribution using the MLE method. The findings demonstrate that the TLE–P distribution provides a superior fit to the datasets compared with the existing distributions considered in this study. Overall, the TLE–P distribution proves useful for modelling positive real-life datasets characterised by positive skewness.

Keywords: TLE–P distribution, Simulation study, Real-life datasets, Maximum Likelihood Estimation (MLE); Maximum Product of Spacing (MPS).

1. Introduction

Real-life datasets occur in various fields of study such as physical science, engineering, biological science, medicine, management, and agriculture, among others, and are mostly skewed; they are modelled with different probability distributions, such as the Topp Leone, Burr X, Frechet, and Pareto distributions, among others, for estimation and statistical inference. However, Pareto exhibits a skewed distribution with heavy or slowly decaying tails (i.e. much of the data is in the tails). The Pareto distribution, named after a Swiss economist, Vilfredo Pareto (1848 - 1923), is defined as a distribution with a shape parameter (also called a slope parameter or Pareto Index) and a location parameter. The distribution has two main applications: first, it is mainly applied to model the distribution of incomes and secondly, the distribution of city populations. However, it is a very familiar probability distribution for modelling and prediction of various socioeconomic aspects. Many applications of the Pareto distribution in actuarial sciences, economics, finance, life testing, climatology, biology and physics have been studied in the literature. Although the distribution has many applications, one major and perhaps most important application of the distribution is in the studies related to the distribution of income. Pareto (1897) initiated this concept in his influential economic texts. Some applications of the distribution in modelling earthquakes, forest fire areas, and oil and gas field sizes are studied by Burroughs and Tebbens (2001). Thus, it can be applied in a variety of situations. For example, it can be applied to model the lifetime of a manufactured item with a certain warranty period.

Several extensions and generalizations of the Pareto distribution have been developed by researchers; such as: Exponentiated Pareto by Stoppa (1990); Gupta *et al*. (1998); the beta-Pareto distribution by Akinsete*et al*. (2008); Exponentiated Pareto distribution by Shawky and Abu-Zinadah (2009); beta generalized Pareto distribution by Mahmoudi (2011); Gamma-Pareto distribution and its application by Alzaatreh *et al.* (2012); Kumaraswamy-Pareto distribution by Bourguignon *et al*. (2013); Kumaraswamy generalized Pareto by Nadarajah and Eljabri (2013), Weibull-Pareto distribution and its applications by Alzaatreh *et al.* (2013); Exponential-Pareto distribution by Al-Kadim and Boshi (2013); transmuted Pareto distribution by Merovci and Puka (2014); Pareto ArcTan by Gómez-Déniz and Calderín-Ojeda (2015); A new Weibull-Pareto distribution by Tahir *et al.* (2016); exponentiated Weibull Pareto by Afify *et al.* (2016); beta transmuted Pareto distribution by Chhetri *et al.*(2017a); Kumaraswamy transmuted Pareto distribution by Chhetri *et al.* (2017b); Pareto-exponential distribution by Waseem and Bashir (2019), cubic transmuted Pareto distribution by Rahman *et al.*(2020) and Pareto weibull distribution by Rana *et al.* (2022).

Therefore, in this paper, a new distribution named the Topp Leone Exponential – Pareto (TLE-P) distribution is developed by extending the Pareto distribution with the Topp Leone Exponential–*G* family of distributions. Thus, the TLE–P distribution represents a significant extension within the broader class of Topp-Leone Exponential-*G* (TLE-G) family of distributions, originally developed by Sanusi *et al.*(2020). The fundamental motivation underlying the construction of the TLE–P distribution is to address the limitations of classical Pareto models when applied to real-world datasets that exhibit complex tail behaviour, varying degrees of skewness, and non-monotonic hazard patterns. This can be done by compounding the Topp-Leone-Exponential generator with a Pareto baseline distribution. The resulting four-parameter model inherits structural flexibility from both Topp-Leone and Exponential families while retaining the heavy-tailed characteristics intrinsic to Pareto-type phenomena. The probability densities of this new TLE–P are defined. Also, some mathematical properties were derived. Simulation studies were conducted to confirm the parameter consistency of this new distribution based on the method of Maximum Likelihood Estimation (MLE) and Maximum Product of Spacing (MPS). The flexibility of the new distribution was illustrated using real-life data sets.

2. Materials and Methods

2.1. Topp Leone Exponential – Pareto Distribution

In this section, the cumulative distribution function (CDF) and probability density function (PDF) of the new distribution, Topp Leone Exponential – Pareto (TLE-P) distribution, are defined.

Now, the CDF and PDF of the Topp Leone Exponential-*G* family of distributions developed by Sanusi *et. al* (2020) are defined in Equations (1) and (2), respectively, as:

$$G_{TLE-G}\left(x;\alpha,\phi,\Omega\right)=\left[1-\exp\left\{-2\alpha\left(\frac{F\left(x;\Omega\right)}{\overline{F}\left(x;\Omega\right)}\right)\right\}\right]^{\phi} \tag{1}$$

and

$$g_{TLE-G}\left(x;\Omega\right)=\frac{2\sigma\lambda f(x,\Omega)}{\left(\overline{G}\left(x;\Omega\right)\right)^{2}}\exp\left\{-2\alpha\left(\frac{F\left(x;\Omega\right)}{\overline{F}\left(x;\Omega\right)}\right)\right\}\left[1-\exp\left\{-2\alpha\left(\frac{F\left(x;\Omega\right)}{\overline{F}\left(x;\Omega\right)}\right)\right\}\right]^{\phi-1} \tag{2}$$

So, let the baseline Pareto distribution be specified in its Type-II (Lomax) form, which is defined on the positive real line and is particularly suitable for hydrological and financial data, where observations are strictly positive and frequently right-skewed. The CDF and PDF of the baseline Pareto are given in Equations (3) and (4), respectively, as:

$$F\left(x\right)=1-\left(\frac{x}{\psi}\right)^{-\mu} \tag{3}$$

and

$$f\left(t\right)=\frac{\mu}{p}\left(\frac{x}{\psi}\right)^{-\mu} \tag{4}$$

where

$\psi,\mu,p>0$, respectively.

Note: Equation (5) is a simplified term of the odd structure, as it is a replaced term after the substitution of both Equations (3) and (4) into (1) and (2), respectively.

$$\frac{1-\left(\dfrac{x}{\theta}\right)^{-\phi}}{\left(\dfrac{x}{\psi}\right)^{-\phi}}=\frac{1}{\left(\dfrac{x}{\psi}\right)^{-\mu}}-1=\left(\frac{x}{\psi}\right)^{\mu}-1 \tag{5}$$

Applying the Topp-Leone Exponential-*G* transformation framework gives the CDF of the new TLE–P distribution and is defined in Equation (6) as:

$$G_{TLE-P}\left(x;\alpha,\phi,\psi,\mu\right)=\left[1-\exp\left\{-2\alpha\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]^{\phi} \tag{6}$$

where

$$x;\alpha,\phi,\psi,\mu>0$$

Differentiating the CDF in Equation (6) with respect to "x" yields the PDF defined in Equation (7) as:

$$g\left(x;\alpha,\phi,\psi,\mu\right)=\frac{2\alpha\phi\left[\frac{\phi}{x}\left(\frac{x}{\psi}\right)^{-\mu}\right]\exp\left\{-2\alpha\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}}{\left[\left(\frac{x}{\psi}\right)^{-\phi}\right]^{2}}\left[1-\exp\left\{-2\alpha\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]^{\phi-1} \tag{7}$$

where

$$x;\alpha,\phi,\psi,\mu>0$$

This closed-form expression is mathematically tractable and satisfies the normalisation condition $\int_{0}^{\infty} f\left(x\right)dx=1$, which has been numerically verified to high precision using adaptive quadrature methods.

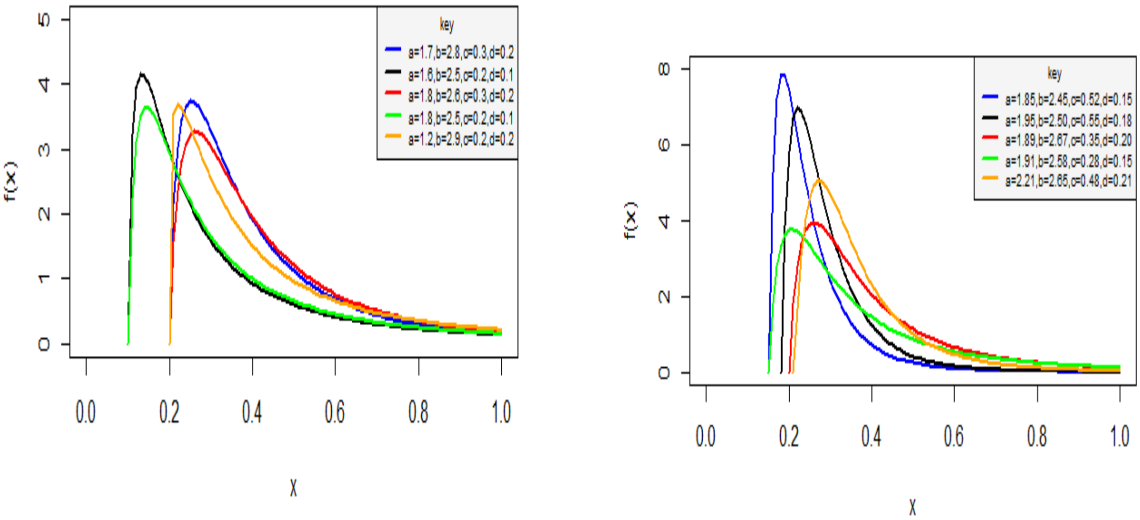

Figure 1. pd function of TLE-P

Figure 1 is the probability density function of the TLE–P distribution showing the nature of the data with a long tail towards the right-hand side of the distribution, which can best be modelled by TEL-P. Hence, a random variable X with density function given in Equation (7) follows TLE-P with survival function, hazard function and cumulative hazard function given in Equation (8), (9) and (10) respectively.

$$S(x,\zeta)=1-\left[1-\exp\left\{-2\alpha\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]^{\phi} \tag{8}$$

$$h(x,\zeta)=\frac{2\alpha\phi\left[\frac{\mu}{p}\left(\frac{x}{\psi}\right)^{-\mu}\right]\exp\left\{-2\alpha\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}}{\left(\frac{x}{\psi}\right)^{2\mu}1-\left[1-\exp\left\{-2\alpha\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]^{\phi}}\left[1-\exp\left\{-2\alpha\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]^{\phi-1} \tag{9}$$

$$H(x,\zeta)=-\ln\left[1-\left[1-\exp\left\{-2\alpha\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]^{\phi}\right] \tag{10}$$

2.2. Linear Expansion of the PDF of TLE -P Distribution

This section introduces a useful expansion for the PDF of the TLE–P distribution. Using the generalised binomial and Taylor series expansions in Equation (11), and if $|x|<1$ and $k>0$ are real non-integers, the power series holds:

$$(1-x)^{k-1}=\sum_{a=0}^{\infty}\frac{(-1)^{a}\Gamma(k)}{a!\Gamma(k-a)}x^{a} \tag{11}$$

Applying the idea of Equation (11) to the terms in (7), this becomes;

$$g_{TLE-P}(x;\alpha,\phi,\psi,\mu)=2\alpha\phi\mu x^{-1}\left(\frac{x}{\psi}\right)^{-\mu}\left[(2\alpha)(a+1)\right]^{b}\left(\left(\frac{x}{\psi}\right)^{\mu}\right)^{b-c}\sum_{a,b,c=0}^{\infty}(-1)^{a+b+c}\frac{b!\Gamma(\alpha)}{a!b!c!(b-c)!\Gamma(\alpha-a)}\left[\left(\frac{x}{\psi}\right)^{-\mu}\right]^{-2}$$

$$=2\alpha\phi\mu x^{-1}\sum_{a,b,c=0}^{\infty}(-1)^{a+b+c}\frac{b!\Gamma(\sigma)\left[(2\phi)(a+1)\right]^{b}}{a!b!c!(b-c)!\Gamma(\alpha-a)}\left(\frac{x}{\psi}\right)^{\mu[1+(b-c)]} \tag{12}$$

Therefore, the PDF of the TLE-P distribution is given as in Equation (13)

$$f_{TLE-P}(x;\alpha,\phi,\psi,\mu)=\sum_{a,b,c}^{\infty}\Omega_{a,b,c}\left(\frac{x}{\psi}\right)^{\mu[1+(b-c)]} \tag{13}$$

where

$$\Omega_{a,b,c}=(-1)^{a+b+c}\frac{2\alpha\phi\mu x^{-1}b!\Gamma(\alpha)\left[(2\phi)(a+1)\right]^{b}}{a!b!c!(b-c)!\Gamma(\alpha-a)}$$

2.3 Mathematical Properties of TLE-P Distribution

This section provides some mathematical properties of the TLE-P distribution such as moments, quantile function, and order statistics.

2.3.1 Moments

Suppose X *is* a random variable with a TLE–P distribution; then the raw moment, $\mu_n^{'}$, is given by

$$\mu_n^{'} = E\left(x^n\right) = \int_{-\infty}^{\infty} x^n g_{\text{TLE-P}}\left(x;\alpha,\phi,\psi,\mu\right) dx, \tag{14}$$

$$\mu_n^{'} = E\left(x^n\right) = x^{n-1} 2\alpha\phi\mu \left(\frac{x}{\psi}\right)^{-\mu} \left(\left(\frac{x}{\psi}\right)^{\mu}\right)^{b-c} \left[\left(\frac{x}{\psi}\right)^{-\mu}\right]^{-2} \sum_{a,b,c=0}^{\infty} (-1)^{a+b+c} \frac{\Gamma(\alpha)\left(2\phi(a+1)\right)^b b!}{a!b!c!\Gamma(\alpha-a)(b-c)!}$$

$$= \sum_{a,b,c=0}^{\infty} (-1)^{a+b+c} \frac{2\alpha\phi\mu x^{n-1}\Gamma(\alpha)\left(2\phi(a+1)\right)^b b!}{a!b!c!\Gamma(\sigma-a)(b-c)!} \left(\frac{x}{\psi}\right)^{\mu[1+(b-c)]} \tag{15}$$

$$\text{Therefore} \quad \mu_n^{'} = E\left(x^n\right) = \sum_{a,b,c=0}^{\infty} \Omega_{a,b,c} \left(\frac{x}{\psi}\right)^{\mu[1+(b-c)]} \tag{16}$$

where

$$\Omega_{a,b,c} = (-1)^{a+b+c} \frac{2\alpha\phi\mu x^{n-1}\Gamma(\alpha)\left(2\phi(a+1)\right)^b b!}{a!b!c!\Gamma(\alpha-a)(b-c)!}$$

2.3.2 Quantile function

The quantile function of the TLE–P distribution defined in Equation (17)is deduced from Equation

(6) above:

$$Q(u) = \psi\left[1 - 2\alpha \log\left(1 - u^{\frac{1}{\phi}}\right)\right]^{\frac{1}{\mu}} \tag{17}$$

2.3.3 Order Statistics

The generalised order statistics for a random variable X that follows the TLE–P distribution are defined in Equation (18) as:

$$f_{i.n} = \frac{g_{\text{TLE-P}}\left(x;\alpha,\phi,\psi,\mu\right)}{\beta(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[G_{\text{TLE-P}}^{j+i-1}\left(x;\alpha,\phi,\psi,\mu\right)\right] \tag{18}$$

So, consider both the CDF and PDF of the TLE–P distribution as defined in Equations (6) and (7), respectively for $f(x)F^{j+i-1}(x)$,

Then

$$g(x)G^{j+i-1}(x)=\frac{2\alpha\phi\left[\frac{\phi}{x}\left(\frac{x}{\psi}\right)^{-\mu}\right]}{\left[\left(\frac{x}{\theta}\right)^{-\phi}\right]^{2}}\sum_{a,b=0}^{\infty}(-1)^{a+b}\frac{\Gamma(\phi(j+i))\left[2\alpha(a+1)\right]^{b}}{a!b!\Gamma(\phi(j+i)-a)}\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)^{b} \quad (19)$$

$$=\sum_{a,b,c=0}^{\infty}(-1)^{a+b+c}\frac{2\alpha\phi\Gamma(\phi(j+i))\left[2\alpha(a+1)\right]^{b}}{a!b!\Gamma(\phi(j+i)-a)}\left[\frac{\phi}{x}\right]\left(\frac{x}{\psi}\right)^{\mu(b-c-1)+2\phi}$$

Therefore, the order statistics of the TLE – P distribution are defined in Equation (20) as:

$$f_{i.n}=\frac{\binom{n-i}{j}}{\beta(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^{j}\sum_{a,b,c=0}^{\infty}(-1)^{a+b+c}\Pi_{a,b} \quad (20)$$

where

$$\Omega_{a,b}=\frac{2\alpha\phi\Gamma(\phi(j+i))\left[2\alpha(a+1)\right]^{b}}{a!b!\Gamma(\phi(j+i)-a)}\left[\frac{\phi}{x}\right]\left(\frac{x}{\psi}\right)^{\mu(b-c-1)+2\phi}$$

2.4 Inference

In this section, two methods of estimation that are going to be utilised, Maximum Likelihood Estimate(MLE)and Maximum Product of Spacing (MPS) to estimate the parameters of the new TLE–P distribution. Also, these two methods are going to be applied to conduct a simulation study to confirm the consistency of the parameter estimates.

2.4.1 Maximum Likelihood Estimate

The maximum likelihood estimation method is the most commonly used method for estimating unknown parameters, among others (Sadiq et al. (2022); Sadiq et al. (2023a); Sadiq et al. (2023b); Sadiq et al. (2022c); Sadiq et al. (2024); Sadiq et al. (2025)). Therefore, the maximum likelihood estimators of the unknown parameters of the TLE–P distribution from complete samples are determined as follows:

Let $x_1, x_2, \ldots, x_n$ be observed values from the TLE–P distribution with a vector of parameters ζ . The Log-likelihood function can be expressed as:

$$l(\zeta)=n\log(2)+n\log(\alpha)+n\log(\phi)+\sum_{i=1}^{\infty}\log\left(\frac{\mu}{x}\left(\frac{x}{\psi}\right)^{-\mu}\right)-2\sum_{i=1}^{\infty}\log\left(\left(\frac{x}{\psi}\right)^{-\mu}\right)-2\phi\sum_{i=1}^{\infty}\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)$$
$$+(\alpha-1)\sum_{i=1}^{\infty}\log\left[1-\exp\left\{2\phi\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right] \quad (21)$$

Differentiating equation (21) with respect to α,ϕ,ψ,μ and equating it to 0, we have:

$$\frac{n}{\alpha}+\sum_{i=1}^{\infty}\log\left[1-\exp\left\{2\phi\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]=0 \quad (22)$$

$$\frac{n}{\phi}-2\sum_{i=1}^{\infty}\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)+\frac{(\alpha-1)\sum_{i=1}^{\infty}\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)}{\left[1-\exp\left\{2\phi\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]}=0 \tag{23}$$

$$\sum_{i=1}^{\infty}\frac{\left(\frac{\mu}{x}\left(\frac{x}{\psi}\right)^{-\mu}\right)^{I}}{\left(\frac{\mu}{x}\left(\frac{x}{\psi}\right)^{-\mu}\right)}-2\sum_{i=1}^{\infty}\frac{\left(\left(\frac{x}{\psi}\right)^{-\mu}\right)^{I}}{\left(\left(\frac{x}{\psi}\right)^{-\mu}\right)}-2\phi\sum_{i=1}^{\infty}\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)^{I}+(\alpha-1)\sum_{i=1}^{\infty}\frac{\left[1-\exp\left\{2\phi\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]^{I}}{\left[1-\exp\left\{2\phi\left(\left(\frac{x}{\psi}\right)^{\mu}-1\right)\right\}\right]}=0 \tag{24}$$

2.4.2 Maximum Product of Spacing

Let $x_1, x_2, \ldots, x_n$ be a random sample from TLE–P distribution having CDF $G(x;\alpha,\phi,\psi,\mu)$. The uniform spacing of a random sample from the TLE–P distribution is defined in Equation (25) as:

$$D_i(x;\alpha,\phi,\psi,\mu) = G(x_i;\alpha,\phi,\psi,\mu)-G\left(x_{(i-1)};\alpha,\phi,\psi,\mu\right) \tag{25}$$

$$=\left[1-\exp\left\{-2\alpha\left(\left(\frac{x_{(i)}}{\psi}\right)^{\mu}-1\right)\right\}\right]^{\phi}-\left[1-\exp\left\{-2\alpha\left(\left(\frac{x_{(i-1)}}{\psi}\right)^{\mu}-1\right)\right\}\right]^{\phi}$$

where

$G(x_0;\alpha,\phi,\psi,\mu)=0$ and $G(x_{n+1};\alpha,\phi,\psi,\mu)=1$. Clearly $\sum_{i=1}^{n+1}D_i(\alpha,\phi,\psi,\mu)=1$

Thus, the MPS estimators $\alpha_{MPS},\phi_{MPS},\psi_{MPS}\,\&\,\mu_{MPS}$ of the parameters $\alpha,\phi,\psi\,\&\,\mu$ are obtained by maximising, $H(\alpha,\phi,\psi,\mu)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\alpha,\phi,\psi,\mu)$ With respect to $\alpha,\phi,\psi\,\&\,\mu$,

The estimators $\alpha_{MPS},\phi_{MPS},\psi_{MPS}\,\&\,\mu_{MPS}$ of the parameters $\alpha,\phi,\psi\,\&\,\mu$ can be obtained by solving the non-linear equations as follows:

$$\frac{d}{d(\alpha,\phi,\psi\,\&\,\mu)}H(\alpha,\phi,\psi\,\&\,\mu)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\phi,\psi,\mu)}\left[G\left(x_{(i)};\alpha,\phi,\psi,\mu\right)-G\left(x_{(i-1)};\alpha,\phi,\psi,\mu\right)\right]=0$$

The individual parameter estimate is obtained by solving the following derivatives as in Equations (26), (27), (28) and (29) respectively:

$$\frac{d}{d(\alpha)}H(\alpha)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\phi,\psi,\mu)}\left[\Delta_\alpha\left(x_{(i)};\alpha,\phi,\psi,\mu\right)-\Delta_\alpha\left(x_{(i-1)};\alpha,\phi,\psi,\mu\right)\right]=0 \tag{26}$$

$$\frac{d}{d(\phi)}H(\phi)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\phi,\psi,\mu)}\left[\Delta_\phi\left(x_{(i)};\alpha,\phi,\psi,\mu\right)-\Delta_\phi\left(x_{(i-1)};\alpha,\phi,\psi,\mu\right)\right]=0 \tag{27}$$

$$\frac{d}{d(\psi)}H(\psi)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\phi,\psi,\mu)}\left[\Delta_\psi\left(x_{(i)};\alpha,\phi,\psi,\mu\right)-\Delta_\psi\left(x_{(i-1)};\alpha,\phi,\psi,\mu\right)\right]=0 \tag{28}$$

$$\frac{d}{d(\mu)}H(\mu)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\phi,\psi,\mu)}\left[\Delta_\mu\left(x_{(i)};\alpha,\phi,\psi,\mu\right)-\Delta_\mu\left(x_{(i-1)};\alpha,\phi,\psi,\mu\right)\right]=0 \tag{29}$$

2.5 Simulation Study

In this section, a simulation study for different sample sizes on the efficiency of MLE and MPS methods to determine the parameters' consistency of the TLE–P distribution is carried out. Thus, a random variable X from the TLE–P distribution is generated using R Software. Samples of size n = 10; 20; 40; 60; 80 and 100 from the TLE–P distribution for some selected combination of parameters are used. This process is repeated1000 times to calculate the mean estimate, mean squared error (MSE) and bias. The results obtained are given in Table 6.1.

From Table 2.1, it is observed that when the sample size increases, the mean of the MLE and MPS approaches the true values. Bias and RMSE decrease towards zero; that is to say, the parameters of the TLE–P distribution are consistent. Therefore, the MLE and MPS methods performed very well in estimating the parameters of the TLE–P distribution.

Table 2.1: Means, Bias and RMSEs for the TLE-P Estimates $(\alpha=1.7,\phi=1.4,\psi=0.6,\mu=1.6)$

		MLE				MPS			
n		α	ϕ	ψ	μ	α	ϕ	ψ	μ
	Mean	1.9560	1.1525	0.6296	1.5995	1.6946	1.4859	0.4085	1.3824
10	Bias	0.2560	-0.2475	0.0296	0.2995	-0.0054	0.0859	-0.1915	0.0824
	RMSE	0.6119	0.5452	0.3516	0.4925	0.4753	0.4588	0.3251	0.3611
	Mean	1.8777	1.2222	0.5298	1.5250	1.7343	1.4490	0.3981	1.3812
20	Bias	0.1777	-0.1778	-0.0702	0.2250	0.0343	0.0490	-0.2019	0.0812
	RMSE	0.4210	0.4406	0.2180	0.3401	0.3546	0.3793	0.2532	0.2532
	Mean	1.8153	1.2681	0.4796	1.5074	1.7406	1.4202	0.4014	1.4080
40	Bias	0.1153	-0.1319	-0.1204	0.2074	0.0406	0.0202	-0.1986	0.1080
	RMSE	0.3135	0.3387	0.1811	0.2853	0.2558	0.3081	0.2287	0.2305
	Mean	1.7973	1.2958	0.4606	1.5080	1.7359	1.4119	0.4033	1.4378
60	Bias	0.0973	-0.1042	-0.1394	0.2080	0.0359	0.0119	-0.1967	0.1378
	RMSE	0.2566	0.2740	0.1743	0.2638	0.2231	0.2502	0.2154	0.2152
	Mean	1.7820	1.3131	0.4519	1.5120	1.7363	1.3985	0.4077	1.4494
80	Bias	0.0820	-0.0869	-0.1481	0.2120	0.0363	-0.0015	-0.1923	0.1494
	RMSE	0.2299	0.2362	0.1740	0.2545	0.1986	0.2218	0.2089	0.2152
	Mean	1.7784	1.3162	0.4485	1.5153	1.7395	1.3878	0.4107	1.4624
100	Bias	0.0784	-0.0838	-0.1515	0.2153	0.0395	-0.0122	-0.1893	0.1624
	RMSE	0.2022	0.2171	0.1713	0.2522	0.1828	0.1937	0.2019	0.2147

3. Results and Discussions

The flexibility of TLE-P in application to real-life datasets is illustrated by comparing the performance with some other existing distributions. The parameters are estimated using MLE only.

The goodness-of-fit statistic and the MLE's for the models' parameters are presented in Tables 3.1 and 3.2. To compare the fitted models, this paper used goodness-of-fit measures and the Akaike information criterion (AIC).The fits of the new TLE–P distribution with other competitive distributions such as Kumaraswamy Pareto (Kw-P), Beta Pareto (BP), Exponential Pareto (E-P), Pareto Weibull (P-W), Pareto Exponential (P-E) and Pareto (P) distributions are analysed and compared using the two data sets

3.1. Dataset 1:

Dataset 1 corresponds to the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place. They were analysed by Choulakian and Stephens (2001). The distribution is highly skewed to the left. However, Akinsete *et al*. (2008), Mahmoudi (2011) and Bourguignon *et al.*(2013) analysed these data using the BP, BGP and Kw-P distributions, respectively. The two datasets were fit to the TLE-P distribution, and the results were compared with the competing models. The required numerical evaluations are implemented using the R software.

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11 .6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0.

Table 3.1**.** Parameter Estimation for Various Distributions for Dataset 1.

Variable	**Estimate**	***-LL***	**AIC**	**RanK**
TLE-P	$\alpha =$ 0.4014 ϕ =0.0025	249.59	507.115	1
	ψ =1.6494 μ =0.9639			
Kw-P	$\theta =$ 2.8553 λ =85.8468	270.2	548.400	2
	θ =0.05280 ϕ =0.1			
BP	α = 3.1473 γ =85.7508	282.0	573.400	3
	θ =0.0088 φ =0.1			
EP	$\psi =$ 2.8797 θ =0.4241	286.1	578.600	4
	ϕ =0.1			
P	$\theta =$ 0.6672 ϕ =3.9606	302.1	608.200	5

Figure 3 displays the hazard function of the TLE-P distribution. The curve exhibits a distinctive U-shaped (bathtub) pattern with high initial hazard, a minimum hazard region between approximately x = 5 and x = 10 and increasing tail hazard beyond x ≈ 10. So, the hazard rate increases monotonically. Also, Figure 4 reveals that the TLE-P density curve (depicted in crimson) tracks the empirical histogram with remarkable fidelity. The model successfully captures the sharp mode near the origin, monotonic decay and the heavy right tail.

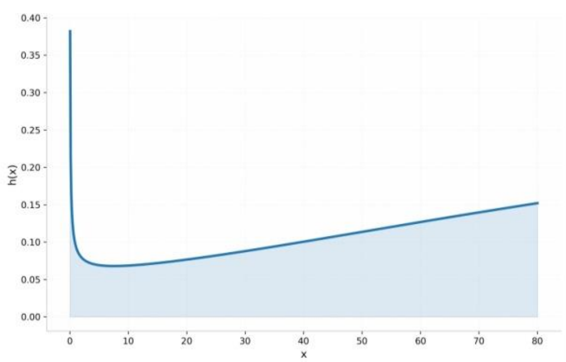

Figure 3:TLE-P Hazard Function using Dataset 1 (bathtub shape).

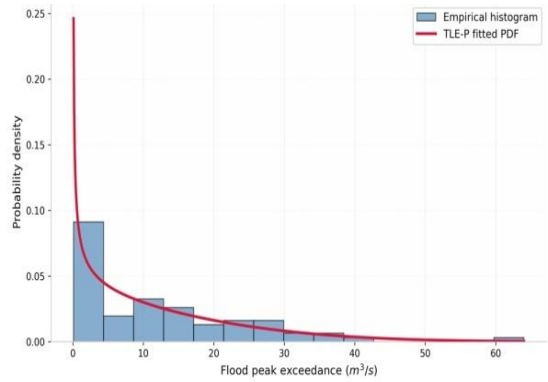

Figure 4:Dataset 1:Wheaton River Flood Peak Exceedances with fitted TLE-P PDF

3.2. Dataset 2:

The study considered dataset 2 of size 39for the Floyd River, located in James, Iowa, USA, which provides the consecutive annual flood discharge rates for the years 1935-1973. The dataset has been previously used by Akinsete *et al.* (2008) and Md. Shohel*et al.* (2022). For the source and details of the data, see Mudholkar and Hutson (1996). The dataset is given as follows:

1460, 4050, 3570, 2060, 1300, 1390, 1720, 6280, 1360, 7440, 5320, 1400, 3240, 2710, 4520, 4840, 8320, 13900, 71500, 6250, 2260, 318, 1330, 970, 1920, 15100, 2870, 20600, 3810, 726, 7500, 7170, 2000, 829, 17300, 4740, 13400, 2940, and 5660.

Table 3.2. Parameter Estimation for Various Distributions for Dataset 1.

Variable	Estimate	*-LL*	AIC	RanK
TLE-P	α = 24.8999 ϕ =1.1101	376.376	760.752	1
	ψ =0.1831 μ =17.7266			
PW	ϕ =1.0850 λ =1.500	377.763	761.526	2
	θ =0.4540			
PE	φ =0.4540 θ =1.5000	380.745	765.491	3
P	ϕ =0.2941 ϖ =36.5483	392.810	787.620	4

The results presented in Tables 3.1 and 3.2 show that the TLE-P distribution has the smallest log-likelihood values (249.59) and (376.376) and AIC values (507.115) and (760.752), respectively and therefore best fits the two datasets compared to other distributions.

Figure 5 presents the hazard function for Dataset 2. It was observed that the hazard exhibits an upside-down bathtub (unimodal) shape: Rapid initial ascent and Monotone decline. After reaching the peak, the hazard decreases steadily as x increases. Also, Figure 6 displays the histogram of Dataset 2 with the superimposed TLE-P density curve based on the MLEs: Several critical observations emerge from this visualisation: extreme peakedness near the origin, Rapid tail decay and Scale adaptability. Thus, the histogram-PDF overlay provides compelling visual evidence that the TLE-P distribution is uniquely capable of simultaneously modelling the extreme peak, the mid-range body, and the heavy upper tail of the datasets. The TLE-P hazard function exhibits a bathtub shape for the Wheaton River data, indicating elevated risk for both minor and catastrophic exceedances with a relatively safe zone at moderate levels. Conversely, for the Floyd River data, the hazard displays an upside-down bathtub shape, signifying that the conditional risk peaks at modest discharge levels before declining. This flexibility in hazard morphology is unattainable by classical Pareto or Beta-Pareto models and underscores the practical utility of the TLE-P distribution in hydrological risk and other assessments."

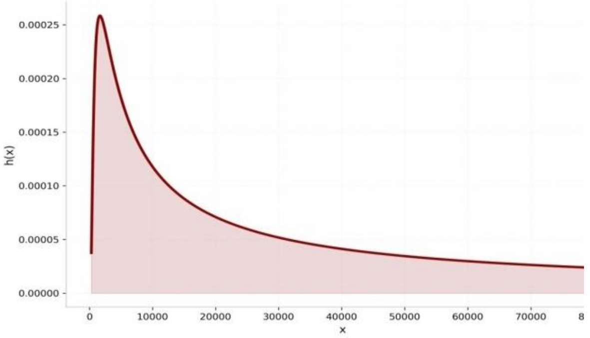

Figure 5: TLE-P Hazard Function using Dataset 2 (upside-down bathtub shape).

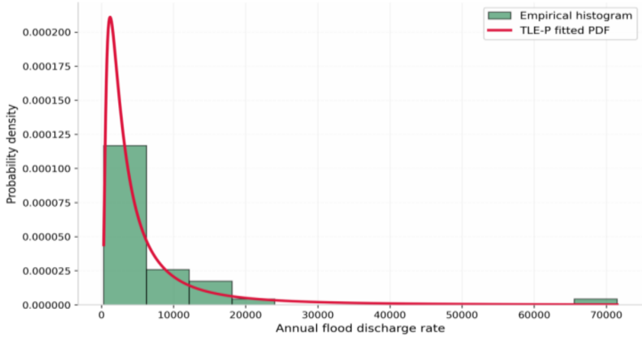

Figure 6: Dataset 2: Floyd River Annual Flood Discharge Rates with fitted TLE-P PDF

5. Conclusion

The Topp–Leone Exponential–Pareto (TLE–P) distribution represents a valuable addition to the toolkit of applied statisticians and hydrologists. The empirical analyses presented in this study demonstrate that the TLE–P density provides both a visually compelling and quantitatively superior fit to the Wheaton River and Floyd River datasets, as evidenced by its close alignment with empirical histograms and its attainment of the minimum Akaike Information Criterion (AIC) values. The hazard function exhibits flexible, non-monotonic shapes that reflect realistic hydrological risk patterns, patterns that are inaccessible to simpler three-parameter competitors. Furthermore, the four-parameter structure is statistically identifiable and computationally tractable, with maximum likelihood estimates (MLEs) and maximum product of spacing (MPS) estimates obtainable via standard numerical optimisation routines.

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