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An Alpha-Beta Power Extension of the New Class Sine-G (ABPNCS) Family of Distributions: Properties and Applications

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Abstract

Classical probability distributions may have limited flexibility for modelling lifetime and survival data characterised by skewness, heavy tails, and non-monotonic hazard rate behaviour. This study aims to develop a flexible Alpha-Beta Power New Class Sine-G (ABPNCS-G) family of distributions and investigate its statistical properties and applicability to lifetime data. The proposed family is constructed by applying alpha-power and beta-power transformations to the New Class Sine-G (NCS-G) family, thereby introducing two additional transformation parameters that enhance the flexibility of the resulting distributions. The cumulative distribution function, probability density function, quantile function, moments, moment generating function, and order statistics are derived, while the unknown parameters are estimated using maximum likelihood estimation with numerical optimisation. As an application, the Alpha-Beta Power New Class Sine-Weibull (ABPNCSW) sub-model is derived and fitted to a bladder cancer remission-time dataset comprising 128 observations. Its performance is compared with the classical Weibull and New Class Sine-Weibull models using information criteria and goodness-of-fit measures, supported by graphical diagnostics. The results show that the ABPNCSW model provides the best overall fit among the competing models, recording the lowest Akaike Information Criterion (AIC) value of 836.4939 and close agreement with the empirical distribution in the density, CDF, and P–P plots. The fitted hazard function also demonstrates a flexible non-monotonic pattern, indicating the model's ability to accommodate varying hazard rate behaviour. The findings demonstrate that the proposed ABPNCS-G family provides a flexible framework for modelling complex lifetime and survival data and offers a useful alternative to existing models. Future studies may extend the family to other baseline distributions and investigate regression and Bayesian modelling frameworks.

Keywords: Alpha-Beta Power Distribution, New Class Sine-G Family, Maximum Likelihood Estimation, Weibull distribution.

1. Introduction

Probability distributions are fundamental to statistical modelling, providing mathematical frameworks for describing uncertainty across diverse fields including reliability engineering, survival analysis, biomedical sciences, environmental studies, finance, and actuarial science. The effectiveness of statistical analyses hinges on the ability of distributions to accurately represent observed data characteristics, making the development of increasingly flexible probability distributions a persistent research priority. Classical lifetime models such as the Exponential, Weibull, Gamma, and Lognormal distributions remain widely used due to their mathematical simplicity and interpretability. However, they are frequently inadequate for contemporary datasets characterised by strong skewness, heavy tails, high kurtosis, and complex hazard behaviours, including bathtub and upside-down bathtub shapes, commonly observed in engineering reliability, medical survival analysis, environmental risk assessment, and industrial quality control. This persistent inadequacy creates an ongoing demand for more flexible statistical frameworks. To address these limitations, numerous distribution-generating techniques have emerged over the past three decades. Marshall and Olkin (1997) pioneered parameter induction methods for extending baseline distributions. Eugene *et al*. (2002) introduced the Beta-G family, substantially broadening attainable distributional shapes, while Cordeiro and de Castro (2011) developed the more tractable Kumaraswamy-G family. Shaw and Buckley (2009) proposed the transmuted family via quadratic rank transmutation, improving skewness and flexibility. More recently, Mahdavi and Kundu (2017) introduced the Alpha-Power transformation, incorporating an additional shape parameter to enhance distributional flexibility.

In recent years, trigonometric-generated families have attracted increasing attention due to their ability to produce flexible density and hazard functions while maintaining relatively simple mathematical forms. Among these, Sapkota *et al*. (2023) proposed the New Class Sine-G (NCS-G) family by applying a sine transformation to an arbitrary baseline CDF. While the NCS-G family demonstrated improved flexibility over many classical and generated models. However, its relatively parsimonious parameterisation may limit its flexibility in accommodating diverse distributional shapes, particularly with respect to skewness, tail behaviour, and hazard-rate patterns. To address this limitation, the proposed ABPNCS-G family introduces two additional transformation parameters through the Alpha–Beta power transformation, providing additional flexibility in adapting the shape and hazard-rate characteristics of the resulting distributions. The proposed family is derived by applying the Alpha-Beta power transformation to the NCS-G family, introducing two additional shape parameters that enhance flexibility while preserving analytical tractability. The additional parameters provide additional flexibility in shaping the distribution, including its skewness, tail behaviour and hazard-rate characteristics, enabling the ABPNCS-G family to model a wider class of lifetime datasets beyond the existing NCS-G framework.

The main contributions of this study are the development and validation of the Alpha-Beta Power New Class Sine-G (ABPNCS-G) family, the derivation of its key distributional properties, estimation of its parameters using maximum likelihood, and development of the ABPNCS-Weibull (ABPNCSW) sub-model for practical application. The remainder of the paper is organised as follows: Section 2 presents the methodology, including the construction of the proposed family, selected distributional properties, maximum likelihood estimation, and derivation of the ABPNCSW sub-model; Section 3 presents the application to the bladder cancer remission-time data and comparison with competing models using goodness-of-fit measures and graphical diagnostics; and Section 4 provides the conclusion and directions for future research.

2. Materials and Methods

2.1. New Class Sine-G (NCS-G) Family of Distributions

In this study, we considered the New Class Sine-G (NCS-G) family of distributions developed by Sapkota et al. (2023). The NCS-G was developed using the T-X approach introduced by Alzaatreh *et. al.* (2013) through transforming the sine function, as defined by Sapkota et al. (2023). The CDF $F(x:\xi)$ of the New Class Sine-G Family of Distribution (NCS-G FD) is defined as:

$$F(x;\xi) = sin\left\{\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right\}; x \in \mathbb{R} \tag{1}$$

with the corresponding PDF given as:

$$f(x;\xi) = \pi \cos\left\{\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right\}\left(\frac{g(x;\xi)}{\left(1+G(x;\xi)\right)^2}\right); x \in \mathbb{R} \tag{2}$$

where: $G(x;\xi)$ and $g(x;\xi)$ are the baseline CDF and PDF, respectively, of a continuous random variable *X*, and $\xi > 0$ is a vector of associated parameters.

2.2. The Proposed Alpha-Beta Power New Class Sine-G Family of Distribution

Although the NCS-G family exhibits improved flexibility over several classical lifetime distributions, its ability to independently regulate skewness, kurtosis, tail behaviour and hazard rate dynamics remains limited due to the absence of additional shape parameters. To overcome this limitation, the Alpha-Beta power transformation is employed. We derive the Alpha-Beta power New Class Sine-G family by applying a two-stage power transformation technique used by Gleaton and Lynch (2006) to transform the NCS-G family.

Adopting the Standard alpha–beta power form in equation 3:

$$T(F) = \frac{F(x)^{\delta}}{F(x)^{\delta} + \left[1-F(x)\right]^{\gamma}} \tag{3}$$

By direct substitution into equation 3, the CDF of the Alpha-Beta power New Class Sine-G family is defined as:

$$F_{\delta,\gamma}(x) = \frac{\left[\sin\left(\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^{\delta}}{\left[\sin\left(\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^{\delta} + \left[1-\sin\left(\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^{\gamma}} \tag{4}$$

where $G(x;\xi)$ the cumulative distribution function (CDF) of the baseline G distribution, x is the observed value of the random variable X and $\xi = (\xi_1, \xi_2 ... \xi_p)$ is the vector of baseline distribution parameters. $\delta > 0$ and $\gamma > 0$ are the alpha-power and beta-power transformation parameters, respectively.

The PDF is derived by differentiating $F_{\alpha,\beta}$ using $f_{NCS} = dF_{NCS} / dx$. Thus, the PDF in closed form is given as:

$$f_{\delta,\gamma}(x) = \frac{\pi \cos\left(\pi \dfrac{G(x;\xi)}{1+G(x;\xi)}\right) \dfrac{g(x;\xi)}{\left(1+G(x;\xi)\right)^2} \left[\delta S^{\delta-1}(1-S)^{\gamma} + \gamma S^{\delta}(1-S)^{\gamma-1}\right]}{\left[S^{\delta} + (1-S)^{\gamma}\right]^2} \tag{5}$$

where $S = \sin\left\{\pi \dfrac{G(x;\xi)}{1+G(x;\xi)}\right\}$

2.3 Selected Distributional Properties

This section presents some important statistical properties of the proposed Alpha-Beta Power New Class Sine-G (ABPNCS-G) family. These properties demonstrate the mathematical flexibility of the proposed model and provide the theoretical basis for its application in reliability analysis, survival modelling, and statistical inference.

2.3.1. Moment of the ABPNCS-G Family

The r^{th} raw moment about the origin for a random variable X following the ABPNCS-G family is derived as follows:

For a distribution with PDF $f(x;\Theta)$, the r^{th} moment is given by:

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x;\Theta) dx \tag{6}$$

where Θ is the vector of parameters for the respective families. Due to the potential complexity of the PDF, these moments are derived using series expansions of the trigonometric components. The first four moments $(\mu'_1, \mu'_2, \mu'_3, \mu'_4)$ are explicitly derived and computed numerically for specific baseline distributions. These moments are fundamental for deriving subsequent measures like variance, skewness, and kurtosis. (Balakrishnan *et al.*, 2004). The PDF of the ABPNCS-G family is given in equation 5; now, by substitution, we derive the r^{th} moment as:

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f_{NCS}(x) \frac{\delta S^{\delta-1}(1-S)^{\gamma} + \gamma S^{\delta}(1-S)^{\gamma-1}}{\left[S^{\delta} + (1-S)^{\gamma}\right]^2} dx \tag{7}$$

where $S = \sin\left\{\pi \dfrac{G(x;\xi)}{1+G(x;\xi)}\right\}$

Using the generalised binomial expansion, we obtain:

$$\mu'_r = \sum_{j=0}^{\infty} (-1)^j (j+1) \int_{-\infty}^{\infty} x^r f_{NCS}(x) \left[\delta S^{-(\delta+1)-\delta j}(1-S)^{\gamma(j+1)} + \gamma S^{-\delta(j+1)}(1+S)^{\gamma(j+1)-1}\right] dx \tag{8}$$

To obtain the first moment (Mean), we set $r = 1$:

$$E(X)=\int_{-\infty}^{\infty} x f_{NCS}(x)\frac{\delta S^{\delta-1}(1-S)^{\gamma}+\gamma S^{\delta}(1-S)^{\gamma-1}}{\left[S^{\delta}+(1-S)^{\gamma}\right]^{2}}dx \tag{9}$$

For the second moment, we set $r=2$:

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f_{NCS}(x)\frac{\delta S^{\delta-1}(1-S)^{\gamma}+\gamma S^{\delta}(1-S)^{\gamma-1}}{\left[S^{\delta}+(1-S)^{\gamma}\right]^{2}}dx \tag{10}$$

Hence, the Variance is:

$$Var(X)=\int_{-\infty}^{\infty} x^2 f_{NCS}(x)\frac{\delta S^{\delta-1}(1-S)^{\gamma}+\gamma S^{\delta}(1-S)^{\gamma-1}}{\left[S^{\delta}+(1-S)^{\gamma}\right]^{2}}dx-\left[\int_{-\infty}^{\infty} x f_{NCS}(x)\frac{\delta S^{\delta-1}(1-S)^{\gamma}+\gamma S^{\delta}(1-S)^{\gamma-1}}{\left[S^{\delta}+(1-S)^{\gamma}\right]^{2}}dx\right]^2 \tag{11}$$

2.3.2 Moment Generating Function (MGF) of the ABPNCS-G Family

The moment generating function, defined as $M_X(t)=E(e^{tX})$, provides an alternative method for deriving moments (Sadiq *et al*. (2022); Sadiq *et al*. (2023a); Sadiq *et al*. (2023b); Sadiq *et al*. (2022c); Sadiq *et al*. (2024); Sadiq *et al*. (2025)). The MGFs for the proposed families were derived and are expressed as:

$$M_X(t)=\int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{12}$$

This function is derived using series expansion techniques to facilitate the term-by-term integration. Thus, the MGF is:

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'_r \tag{13}$$

Recall the PDF of the ABPNCS-G family expressed in equation 5; by substituting the PDF into equation 11, we obtain:

$$M_{ABPNCS}(t)=\int e^{tx} f_{NCS}(x)(1-F_{NCS}(x))^{\delta-1}\left[1-(1-F_{NCS}(x))^{\delta}\right]^{\gamma-1}dx$$

By applying the binomial expansion, substituting the CDF and expanding, e^{tx} we obtain:

$$M_{ABPNCS}(t)=\sum_{j=0}^{\infty}(-1)^j\,(j+1)\sum_{r=0}^{\infty}\frac{t^r}{r!}\begin{bmatrix}\delta\int x^r f_{NCS}(x)S^{-(\delta+1)-\delta j}(1-S)^{\gamma(J+1)}\\+\gamma\int x^r f_{NCS}(x)S^{-\delta(j+1)}(1-S)^{\gamma(J+1)-1}dx\end{bmatrix} \tag{14}$$

where $S=\sin\left\{\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right\}$

In general, the derived Moment Generating Functions (MGFs) of the proposed family of distributions can be differentiated with respect to t to recover all respective moments of the families, thereby obtaining the variance, skewness and kurtosis of the proposed families.

2.3.3. Order Statistics of the ABPNCS-G family

The PDF of the k^{th} order statistic, $X_{(k)}$, from a random sample of size n drawn from the proposed distributions were derived. As described by David and Nagaraja (2004); Sadiq *et al*. (2022); Sadiq *et al*. (2023a); Sadiq *et al*. (2023b); Sadiq *et al*. (2022c); Sadiq *et al*. (2024); Sadiq *et al*. (2025), the general form is:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)\left[F(x)\right]\left[1-F(x)\right]^{n-k} \tag{15}$$

where $F(x)$ and $f(x)$ are the CDF and PDF of the ABPNCS-G family, respectively. Particular focus was given to the distributions of the minimum $X_{(1)}$ and maximum $X_{(n)}$ order statistics. Recall the PDF and CDF of the NCS-G families expressed in equations 1 and 2. By substituting the CDF in equation (4) and PDF in equation (5) of the ABPNCS-G family into equation (11), we obtain:

$$f_{X_{(k)}}^{ABPNCS}(x) = \frac{n!}{(k-1)!(n-k)!} f_{NCS}(x) \frac{\left[\delta S^{\delta-1}(1-S)^{\gamma} + \gamma S^{\delta}(1-S)^{\gamma-1}\right] S^{\delta(k-1)}(1-S)^{\gamma(n-k)}}{\left[S^{\delta}+(1-S)^{\gamma}\right]^{n+1}} \tag{16}$$

For the minimum order statistics, $k=1$:

$$f_{X_{(1)}}^{ABPNCS}(x) = n f_{NCS}(x) \frac{\left[\delta S^{\delta-1}(1-S)^{\gamma} + \gamma S^{\delta}(1-S)^{\gamma-1}\right](1-S)^{\gamma(n-1)}}{\left[S^{\delta}+(1-S)^{\gamma}\right]^{n+1}} \tag{17}$$

For the maximum order statistics, $k=n$:

$$f_{X_{(n)}}^{ABPNCS}(x) = n f_{NCS}(x) \frac{\left[\delta S^{\delta-1}(1-S)^{\gamma} + \gamma S^{\delta}(1-S)^{\gamma-1}\right] S^{\delta(n-1)}}{\left[S^{\delta}+(1-S)^{\gamma}\right]^{n+1}} \tag{18}$$

The order statistics obtained for the proposed family can be used to obtain sample extremes, reliability measures and asymptotic properties of the proposed families.

2.4 Parameter Estimation

2.4.1. Maximum Likelihood Estimation (MLE) for the ABPNCS-G Family

The Maximum Likelihood Estimation method was used for parameter estimation due to its desirable asymptotic properties, such as consistency and normality. For a random sample $x_1, x_2, \ldots, x_n$, a family of distributions with parameter Θ, the likelihood function is:

$$L(\Theta) = \prod_{i=1}^{n} f(x_i;\Theta) \tag{19}$$

And the log-likelihood function is $\ell(\Theta) = \log L(\Theta)$. The maximum likelihood estimates (MLEs) are obtained by solving the system of equations derived from the partial derivatives of the log-likelihood function with respect to each parameter in Θ, Aldrich (1997). Given the potential complexity, the solutions were obtained numerically using iterative optimisation algorithms in R statistical software. Recall the CDF in equation (4) and PDF in equation (5) of the ABPNCS-G family. Assuming $x_1, x_2, \ldots, x_n$ is a random sample from the ABPNCS-G family, the likelihood function is:

$$L(\Theta) = \prod_{i=1}^{n} f_{ABPNCS}(x_i)$$

Where $\Theta = (\delta, \gamma, \xi)$ with ξ denoting the parameter vector of the baseline NCS-G family. By substituting the density into the likelihood function, we obtain:

$$L(\Theta) = \prod_{i=1}^{n} f_{NCS}(x_i) \prod_{i=1}^{n} \frac{\delta S_i^{\delta-1}\left(1-S_i\right)^{\gamma} + \gamma S_i^{\delta}\left(1-S_i\right)^{\gamma-1}}{\left[S_i^{\delta} + \left(1-S_i\right)^{\gamma}\right]^2} \tag{20}$$

where $S_i = S\left(x_i;\xi\right)$

By taking logarithms, we obtain the log-likelihood function, $\ell(\Theta) = \log L(\Theta)$, as:

$$\begin{aligned}\ell(\Theta) = &\sum_{i=1}^{n} \ln f_{NCS}\left(x_i\right) + \sum_{i=1}^{n} \ln\left[\delta S_i^{\delta-1}\left(1-S_i\right)^{\gamma} + \gamma S_i^{\delta}\left(1-S_i\right)^{\gamma-1}\right] \\ &-2\sum_{i=1}^{n} \ln\left[S_i^{\delta} + \left(1-S_i\right)^{\gamma}\right]\end{aligned} \tag{21}$$

To obtain the score function for γ, we differentiate each term of the log-likelihood in equation (20) with respect to γ, thus:

$$\begin{aligned}U_{\gamma} = \frac{\partial \ell}{\partial \gamma} = &\sum_{i=1}^{n} \frac{\delta S_i^{\delta-1}\left(1-S_i\right)^{\gamma} \ln\left(1-S_i\right) + S_i^{\delta}\left(1-S_i\right)^{\gamma-1}\left[1+\gamma \ln\left(1-S_i\right)\right]}{\delta S_i^{\delta-1}\left(1-S_i\right)^{\gamma} + \gamma S_i^{\delta}\left(1-S_i\right)^{\gamma-1}} \\ &-2\sum_{i=1}^{n} \frac{\left(1-S_i\right)^{\gamma} \ln\left(1-S_i\right)}{S_i^{\delta} + \left(1-S_i\right)^{\gamma}}\end{aligned} \tag{22}$$

Similarly, to obtain the score function for δ, we differentiate each of the terms in the likelihood function in equation (20) w.r.t δ, hence:

$$U_{\delta} = \frac{\partial \ell}{\partial \delta} = \sum_{i=1}^{n} \frac{S_i^{\delta-1}\left(1-S_i\right)^{\gamma}\left[1+\delta \ln S_i\right] + \gamma S_i^{\delta}\left(1-S_i\right)^{\gamma-1} \ln S_i}{S_i^{\delta-1}\left(1-S_i\right)^{\gamma} + \gamma S_i^{\delta}\left(1-S_i\right)^{\gamma-1}} - 2\sum_{i=1}^{n} \frac{S_i^{\delta} \ln S_i}{S_i^{\delta} + \left(1-S_i\right)^{\gamma}} \tag{23}$$

To obtain the score function for the baseline parameters ξ_j, we differentiate w.r.t each of the parameters ξ_i using the chain rule: $\frac{\partial S_i}{\partial \xi_j} = \pi \cos\left(\pi \frac{G_i}{1+G_i}\right) \frac{g_i^{(j)}}{\left(1+G_i\right)^2}$

Thus:

$$U_{\xi_j} = \sum_{i=1}^{n} \frac{\partial}{\partial \xi_j} \ln f_{NCS}\left(x_i\right) + \sum_{i=1}^{n} \frac{\partial \ln R\left(S_i\right)}{\partial S_i} \frac{\partial S_i}{\partial \xi_j} \tag{24}$$

Where $R\left(S\right) = \dfrac{\alpha S^{\alpha-1}\left(1-S\right)^{\beta} + \beta S^{\alpha}\left(1-S\right)^{\beta-1}}{\left(S^{\alpha} + \left(1-S\right)^{\beta}\right)^{2}}$

For each baseline parameter, $j = 1, 2, \ldots, p$, setting $\dfrac{\partial \ell}{\partial \xi_j} = u_{\xi_j} = 0$, yields the system of likelihood equations. Therefore, the score vector for the ABPNCS-G family is: $U(\Theta) = \left(U_{\delta}, U_{\gamma}, U_{\xi_2}, \ldots, U_{\xi_p}\right)^{T}$

Thus, setting it to zero yields the system of nonlinear equations whose solution gives the maximum likelihood estimators for the ABPNCS-G parameters. Because closed-form solutions generally do not exist, the obtained parameter estimates are solved numerically using iterative optimisation procedures that are implemented in the R statistical software.

2.5. ABPNCS-Weibull Distribution

To demonstrate the flexibility and practical applicability of the proposed Alpha-Beta Power New Class Sine-G family, a special sub-model is developed by selecting the Weibull distribution as the baseline distribution. Embedding the Weibull distribution within the proposed ABPNCS-G framework further enhances its flexibility by introducing two additional shape parameters capable of accommodating a wider range of distributional behaviours. The two-parameter Weibull distribution with parameters $\alpha, \beta > 0$, with PDF:

$$f\left(x;\alpha,\beta\right) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^{\alpha}}; x > 0, \alpha, \beta > 0 \tag{25}$$

and CDF:

$$F\left(x;\alpha,\beta\right) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}; x > 0 \tag{26}$$

By substituting the CDF of the Weibull distribution in equation (25) into the CDF of the NCS-G family, we get the NCS-Weibull CDF as:

$$F_{NCSW}(x) = sin\left[\pi\left(\frac{1-e^{-(x/\beta)^{\alpha}}}{2-e^{-(x/\beta)^{\alpha}}}\right)\right]; x \in \mathbb{R} \tag{27}$$

Likewise, for the PDF, we substitute the PDF of the Weibull distribution into the PDF of the NCS-G family expressed in equation (1)

$$f_{NCSW}(x;\xi) = \pi \cos\left[\frac{\pi(1-e^{-(x/\beta)^{\alpha}}}{2-e^{-(x/\beta)^{\alpha}}}\right]\left[\frac{\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^{\alpha}}}{\left(2-e^{-(x/\beta)^{\alpha}}\right)^{2}}\right]; x, \alpha, \beta > 0 \tag{28}$$

Recall the CDF of the ABPNCS-G family expressed in equation (4); we substitute the CDF of the NCSW in equation (27) to obtain:

$$F_{ABPNCSW}(x)=\frac{\left[\sin\left(\pi\dfrac{1-\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]}{2-\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]}\right)\right]^{\delta}}{\left[\sin\left(\pi\dfrac{1-\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]}{2-\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]}\right)\right]^{\delta}+\left[1-\sin\left(\pi\dfrac{1-\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]}{2-\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]}\right)\right]^{\gamma}} \tag{29}$$

Where: $\alpha,\beta>0$ are the shape and scale parameters, respectively; $\delta,\gamma>0$ are the alpha-power and beta-power transformation parameters, respectively; and $x>0$ is the observed value of the random variable X. Likewise, for the PDF of the ABPNCS-Weibull distribution, we substitute the PDF in equation (28) and CDF in equation (27) of the NCSW into the PDF of the ABPNCS-G family as:

$$f_{ABPNCSW}(x)=\frac{\pi\cos\left(\pi\dfrac{1-\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]}{2-\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]}\right)}{\left(2-\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]\right)^{2}}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]\frac{\delta S(x)^{\delta-1}(1-S(x))^{\gamma}+\gamma S(x)^{\delta}(1-S(x))^{\gamma-1}}{\left[S(x)^{\delta}+(1-S(x))^{\gamma}\right]^{2}} \tag{30}$$

Where $S(x)=sin\left[\pi\left(\dfrac{1-e^{-(x/\beta)^{\alpha}}}{2-e^{-(x/\beta)^{\alpha}}}\right)\right]$

3. Results and Discussions

The theoretical development presented above establishes the ABPNCS-Weibull distribution as a flexible extension of the classical Weibull model. To evaluate its practical performance, the proposed sub-model is fitted to real lifetime datasets and compared with some existing competing distributions using standard goodness-of-fit measures. The results of this empirical analysis are presented in this section. The 128-patient bladder cancer remission-time dataset originally sourced from Lee and Wang (2003) was used for this study, along with a summary of descriptive statistics and applications to some selected extensions of the Weibull distribution. We have compared the performance of the Alpha-Beta Power New Class Sine-G Weibull distribution (*ABPNCS-W*) to those of the New-Class Sine-G Weibull (*NCSW*) and the classical Weibull distribution (*WD*). We evaluated the performance of the above-listed models using the *AIC* (Akaike Information Criterion). It is considered that the model with the smallest value of AIC was chosen as the best model to fit the data. The data are: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 0.90, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 1.05, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.19, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.26, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.35, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.40, 2.87, 5.62, 7.87, 11.64, 17.36, 1.46, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.76, 3.25, 4.40, 5.85, 8.26, 11.98, 19.13, 2.02, 3.31, 4.50, 6.25, 8.37, 12.02, 20.28, 2.02, 3.36, 4.51, 6.54, 8.53, 12.03, 21.73, 2.07, 3.36, 6.76, 12.07, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. It's summarised as follows:

Table 1: Summary Statistics for the bladder cancer remission-time dataset

Parameters	**N**	**Min.**	**Median**	**Mean**	**Max.**	**Variance**	**Skewness**	**Kurtosis**
Values	128	0.08	6.64	9.37	79.05	95.57	3.33	15.86

Table 2. The performance of the selected models using the *AIC* value of the models evaluated at the *MLEs* based on the bladder cancer remission-time dataset

Distributions	**Parameter estimates (standard errors)**	**Negative log-likelihood ($-\ell$)**	***AIC***	**Ranks**
ABPNCSW	$\beta = 516.3989(704.2903)$ $\gamma = 4.6416(2.4023)$ $\delta = 2.4981(1.0983)$ $\alpha = 0.4141(0.1174)$	414.2470	836.4939	1
NCSW	$\beta = 24.8688(2.3386)$ $\alpha = 1.1218(0.0718)$	417.2028	838.4055	2
Weibull	$\beta = 9.4458(0.8387)$ $\alpha = 1.0449(0.0668)$	419.1582	842.3164	3

The bladder cancer remission-time dataset demonstrates the practical advantage of the proposed ABPNCSW distribution. While the classical Weibull and New Class Sine Weibull (NCSW) models adequately describe the data, the proposed ABPNCSW model achieved the best overall fit according to the information criteria. This superior performance can be attributed to the additional alpha and beta transformation parameters, which provide additional flexibility in modelling skewness and tail behaviour commonly observed in medical survival data. Importantly, compared to the NCSW model, the ABPNCSW achieves a 1.91-point improvement in AIC (838.4055 vs. 836.4939), demonstrating the benefit of the additional parameters. Although the ABPNCSW model attained the lowest AIC, the large condition number of the observed Hessian indicates numerical ill-conditioning, suggesting that inference on some parameter estimates should be interpreted cautiously.

The distributional functions presented in Figure 1 (PDF, CDF, Survival Function, and Hazard Rate) demonstrate the flexibility of the proposed ABPNCSW distribution in modelling the bladder cancer remission data. The fitted hazard function exhibits a non-monotonic pattern, with an initial increase followed by a decline. Statistically, this indicates that the instantaneous event rate is not adequately represented by a constant or strictly monotonic hazard structure. The result illustrates the ability of the ABPNCSW model to accommodate non-monotonic hazard behaviour in the observed remission-time data.

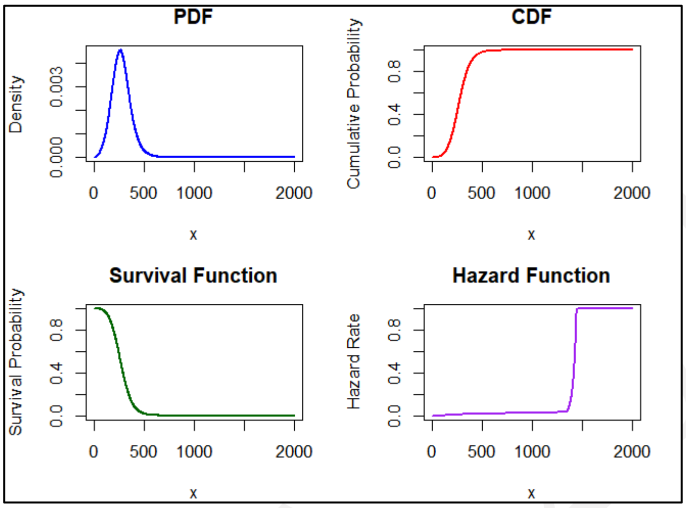

Figure 1: Set of distributional functions of the ABPNCSW distribution

Figure 2 presents the fitted probability density functions of the Weibull, NCSW, and ABPNCSW distributions for the bladder cancer remission-time data. All three models exhibit a positively skewed, unimodal shape, with the densities rising rapidly to a peak at relatively short remission times and subsequently declining toward zero as remission time increases. The three curves are generally similar over the central and upper portions of the distribution, indicating comparable representation of the long right tail. However, the ABPNCSW curve (red) shows a slightly shifted and more pronounced peak relative to the Weibull (blue) and NCSW (green) curves, suggesting additional flexibility in representing the concentration of observations at shorter remission times. The differences among the fitted curves become smaller beyond approximately 10–20 months, where all models provide a similar gradual decline. Overall, the figure demonstrates that the three distributions capture the principal skewed and right-tailed structure of the remission-time data, while the ABPNCSW provides additional flexibility in the shape of the fitted density.

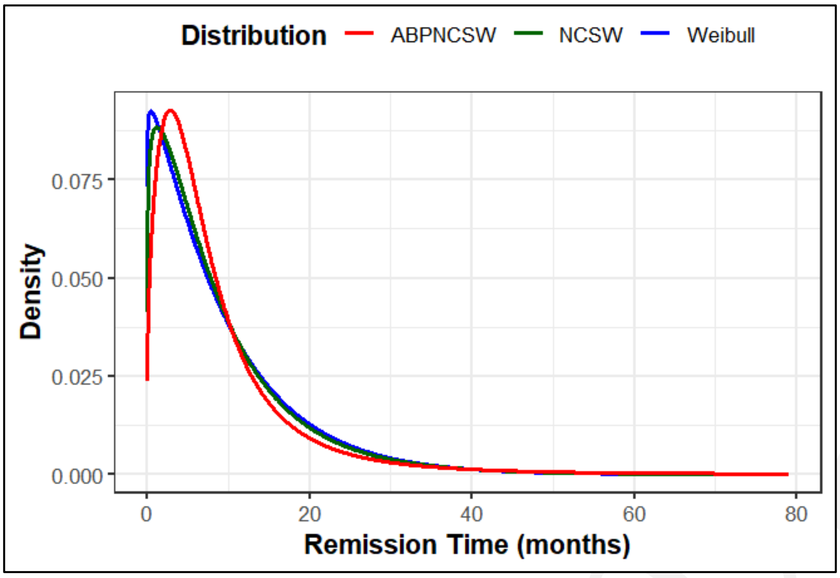

Figure 2. Bladder Cancer Remission Times - Fitted Densities (n = 128)

Figure 3 presents the empirical cumulative distribution function (CDF) together with the fitted CDFs of the proposed ABPNCSW, the NCSW and classical Weibull distributions. The CDF plot shows that the Weibull, NCSW, and ABPNCSW distributions provide broadly similar cumulative probability patterns for the bladder cancer remission-time data. All three curves increase rapidly at shorter remission times, indicating that a large proportion of the observed cumulative probability is accumulated within the early-to-intermediate remission period. The curves then gradually level off and approach 1 as remission time increases, reflecting the right-skewed nature of the data and the relatively small number of observations at longer remission times. The ABPNCSW curve (red) differs slightly from the Weibull (blue) and NCSW (green), particularly in the lower-to-middle range, but the three curves become very close beyond approximately 20–30 months. Overall, the proximity of the fitted CDFs suggests that all three models capture the broad cumulative structure of the remission-time data reasonably well; therefore, differences in model adequacy are more appropriately assessed using the numerical goodness-of-fit measures shown in Table 2.

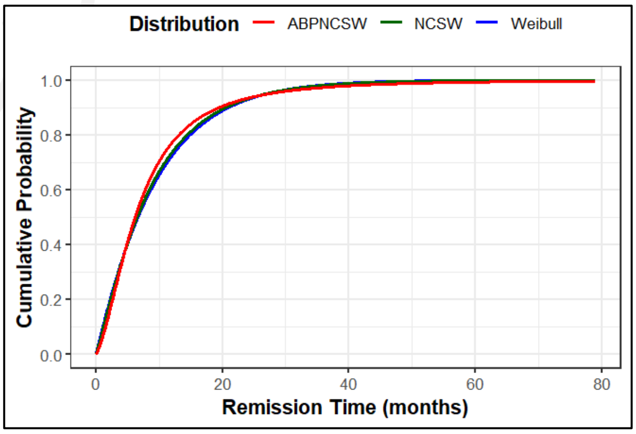

Figure 3. Empirical vs Fitted CDFs - Bladder Cancer Remission Times

Figure 4 presents the P–P plot comparing the theoretical probabilities from the fitted ABPNCSW, NCSW, and Weibull distributions with the corresponding empirical probabilities for the bladder cancer remission-time data. All three sets of points generally follow the 45∘ reference line, indicating good overall agreement between the fitted and empirical distributions. The ABPNCSW points (red) show close alignment with the reference line across most of the probability range, although some deviations are visible around the middle-to-upper probabilities, approximately 0.5–0.8. The NCSW (green) and Weibull (blue) points also remain relatively close to the diagonal, with modest departures in the same region. The differences among the three models are relatively small, particularly in the lower probability range and toward the upper end of the distribution. Overall, the P–P plot suggests that all three models provide reasonable representations of the observed remission-time distribution, while the relatively close alignment of the ABPNCSW points with the reference line is consistent with its favourable goodness-of-fit performance.

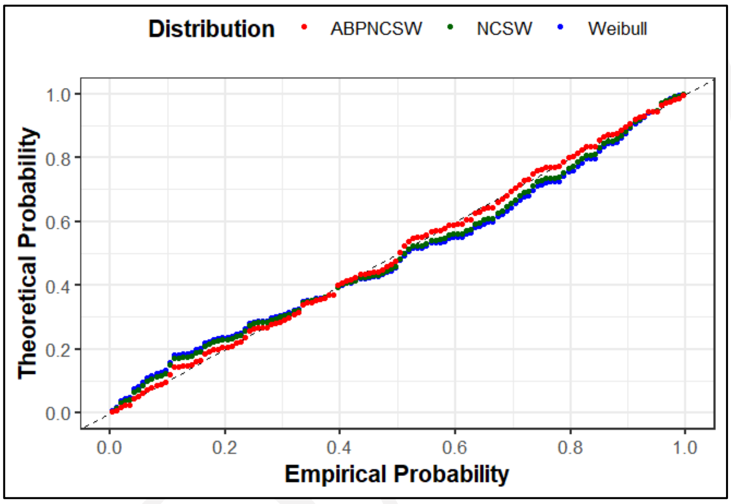

Figure 4. P-P Plot - Comparison of Fitted Distributions

4. Conclusion

This study introduced the Alpha-Beta Power New Class Sine-G (ABPNCS-G) family of distributions, a novel and flexible class of probability models obtained by applying the Alpha–Beta power transformation to the New Class Sine-G (NCS-G) family. The additional transformation parameters increase the flexibility of the resulting family and allow a wider range of distributional and hazard-rate shapes. Several important statistical properties of the family were derived, including the cumulative distribution function, probability density function, quantile function, moments, moment generating function, and order statistics. The unknown parameters were estimated using the maximum likelihood method through numerical optimisation implemented in R.

The practical applicability of the proposed family was demonstrated through its Weibull sub-model, the ABPNCSW distribution, using a bladder cancer remission dataset comprising 128 observations. The empirical results showed that the ABPNCSW model achieved the lowest AIC among the models considered, including the classical Weibull and New Class Sine-Weibull distributions, indicating the most favourable information-criterion-based fit for this dataset. In addition, graphical diagnostics, including the fitted density, cumulative distribution, and P–P plots, showed close agreement between the proposed model and the observed data, indicating its ability to effectively capture the positive skewness and tail behaviour of the remission-time distribution. Overall, the results demonstrate that the ABPNCS-G family provides a flexible and useful framework for modelling complex lifetime and survival data. The empirical illustration is based on a single bladder cancer remission-time dataset, and therefore the observed improvement in fit should not be interpreted as universal superiority of the proposed model. In addition, the numerical estimation of the four-parameter ABPNCSW model exhibited some degree of ill-conditioning, indicating that parameter-level inference should be interpreted with caution. Further simulation studies and applications to additional lifetime datasets are therefore required to assess estimator stability and the general applicability of the proposed family. Future research may consider extending the proposed family to other baseline distributions, developing regression models for censored survival data, investigating Bayesian estimation procedures, and exploring applications in reliability engineering, actuarial science, environmental studies, and other areas involving lifetime data.

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