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On the Parameter Estimation of Marshall-Olkin Inverse Power Logistic-Exponential Distribution in the Presence of Right Censoring

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Abstract

Accurate parameter estimation for lifetime distributions is fundamental in survival analysis, particularly when observations are subject to right censoring, as incomplete data can reduce the efficiency of statistical inference. This study estimates the parameters of a new distribution called the Marshall-Olkin inverse power Logistic-Exponential distribution using the maximum likelihood estimation (MLE) technique. The estimation is carried out in the presence of right censoring. To facilitate interval estimation and statistical inference, the Fisher Information Matrix was derived from the observed likelihood function. Monte Carlo simulation study involving 1,000 replications and sample sizes of 30, 50, 75, 100, and 200 was conducted to evaluate the finite-sample performance of the MLE. The simulation results showed that the MLE is consistent, with estimation accuracy improving as the sample size increases. On the other hand, the standard errors, biases and root mean square errors decrease as sample size increases. In addition, the leukaemia dataset was used in fitting the Marshall-Olkin inverse power Logistic-Exponential distribution using the aforementioned estimation procedure and compared its performance with that of some existing Marshall-Olkin family of distributions. The results showed that the Marshall-Olkin inverse power Logistic-Exponential distribution provided the best overall fit based on the log-likelihood (-64.87067), Akaike Information Criterion (AIC=137.7413), Bayesian Information Criterion (BIC=147.7406), and Consistent Akaike Information Criterion (CAIC=151.7406). This showed that the Marshall-Olkin inverse power Logistic-Exponential distribution is more flexible and suitable in modelling the leukaemia dataset.

Keywords: Parameter Estimation, Marshall-Olkin family, Inverse Power Logistic-Exponential distribution, right-censored data, survival analysis, Maximum Likelihood Estimation.

1. Introduction

Lifetime distributions are widely used in survival analysis and medical research to model the time until the occurrence of an event of interest (Karim & Islam, 2019; Sadiq *et al*., 2025). In practice, survival data often contain right-censored observations because some subjects do not experience the event before the end of the study or are lost to follow-up (Kotian *et al*., 2026). Consequently, flexible lifetime distributions are essential for accurate statistical inference (Brito *et al.,* 2024). The fundamental aspect of statistical modelling is parameter estimation because the usefulness of any probability distribution depends on the accuracy with which its unknown parameters can be estimated (Behnke & Moneta, 2013).

Parameter estimation is one of the most important aspects of statistical modelling because the practical usefulness of any probability distribution depends largely on how accurately its unknown parameters can be estimated (Van den Bos, 2007). Several estimation procedures have been proposed in the statistical literature, including the method of moments, least squares estimation, weighted least squares estimation, percentile estimation, Bayesian estimation and maximum likelihood estimation (MLE). Among these methods, MLE remains the most widely adopted because, under suitable regularity conditions, it produces consistent estimators, asymptotically unbiased, asymptotically efficient and asymptotically normally distributed (Zhong & Yuan, 2011). Gupta and Kundu (2000) compared several estimation methods for the generalized Exponential distribution and demonstrated that MLE possesses desirable finite-sample and asymptotic properties. Hossain (2018) investigated parameter estimation for the generalised exponential distribution and showed that MLE provides reliable and efficient estimates, particularly as the sample size increases. Furthermore, Al-Salafi and Adham (2018) successfully employed MLE to estimate the parameters of the Odd Generalised Exponential-Exponential distribution, demonstrating its effectiveness through Monte Carlo simulation and real-life data applications.

Classical lifetime distributions such as the exponential and Weibull distributions are commonly used in survival analysis due to their simplicity. However, these distributions may not adequately describe complex survival data exhibiting different hazard rate behaviours. This has led to the development of more flexible lifetime models. One such model is the Inverse Power Logistic-Exponential (IPLE) distribution proposed by Al Sobhi (2020), which provides greater flexibility than the Logistic-Exponential distribution developed by Lan and Leemis (2008) in modelling lifetime data.

The Marshall-Olkin method introduced by Marshall and Olkin (1997) is a popular technique for extending existing lifetime distributions by incorporating an additional parameter that improves their flexibility in modelling density, survival and hazard rate functions. In the literature, existing studies show that many baseline lifetime models developed through inverse-power and related transformations can either be extended or have already been extended through the Marshall-Olkin scheme. For example, Babagana *et al*. (in press) developed Marshall-Olkin Inverse Power Logistic-Exponential (MOIPLE) distribution, Osi *et al*. (2024) introduced the Marshall-Olkin Cosine Topp-Leone-G family of distribution**s**, Hassan and Nassr (2021) proposed the Marshall-Olkin Inverse Power Lomax (MOIPL) distribution, Jose *et al*. (2014), extended Exponential distribution, Ristić and Kundu (2015) extended Generalized Exponential distribution, Nassar *et al*. (2019) extended Alpha Power Exponential distribution, and Mansoor *et al*. (2019) developed Logistic-Exponential distribution, Marshall-Olkin Extended Log-Logistic distribution by Gui (2013) and Marshall-Olkin Half Logistic distribution by Yeğen and Berred (2018).

In summary, the aim of this study is to investigate the maximum likelihood estimation procedure of the parameters of the Marshall-Olkin Inverse Power Logistic-Exponential (MOIPLE) distribution developed by Babagana *et al*. (in press).

2. Methodology

Babagana *et al.* (in press) introduced the MOIPLE distribution as a lifetime distribution with four parameters. The probability density function (pdf) is defined by:

$$f(x) = \frac{\theta\alpha\beta\lambda x^{-(\beta+1)} e^{\lambda x^{-\beta}} \left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha-1}}{\left[1+\theta\left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha}\right]^2} \tag{1}$$

where $x > 0$; α, β and $\theta > 0$ are the shape parameters, $\lambda > 0$ is the scale parameter, and the cumulative distribution function (cdf), survival function and hazard rate function of the corresponding MOIPLE distribution defined in (1) are respectively given as:

$$F(x) = \frac{1}{1+\theta\left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha}} \tag{2}$$

$$S(x) = \frac{\theta\left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha}}{1+\theta\left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha}}, \tag{3}$$

and

$$h(x) = \frac{\alpha\beta\lambda x^{-(\beta+1)} e^{\lambda x^{-\beta}}}{\left(e^{\lambda x^{-\beta}} - 1\right)\left[1+\theta\left(e^{\lambda x^{-\beta}} - 1\right)^{\alpha}\right]} \tag{4}$$

The plots of the pdf and hazard rate function (hrf) of the MOIPLE distribution are shown in Figures 1 and 2, respectively.

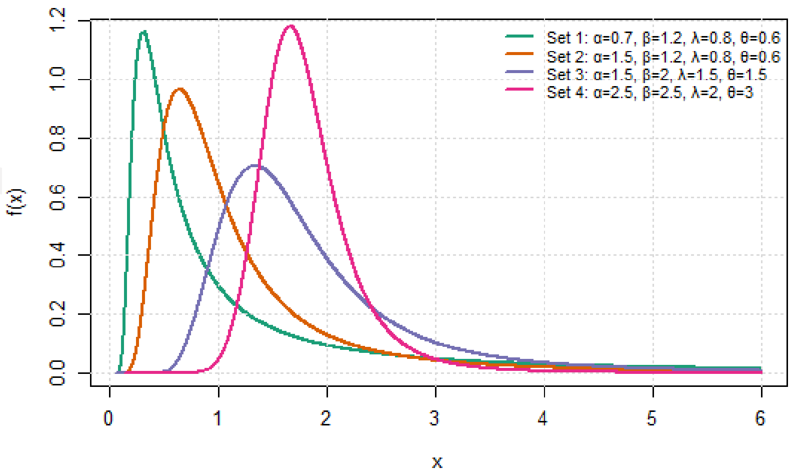

Figure 1. PDF plots of MOIPL distribution for some parameter values

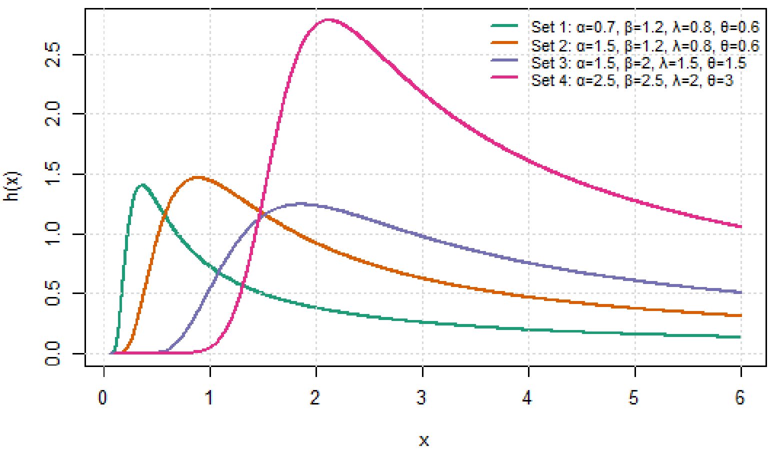

Figure 2. Hazard rate function plots of the MOIPLE distribution for some parameter values

The PDF of the MOIPLE distribution shown in Figure 1 exhibits considerable flexibility under different parameter combinations. The first parameter setting produces a density that is positively skewed with its peak occurring close to the origin, indicating that most observations are concentrated at smaller lifetime values. Such behaviour is appropriate for systems that experience relatively early failures. As the parameter values are varied, the density becomes moderately peaked and gradually shifts towards larger lifetime values. This demonstrates that the MOIPLE distribution is capable of modelling lifetimes whose failure times occur at different stages of operation.
The hazard rate function in Figure 2 illustrates the instantaneous risk of failure at any given time. The generated hazard curves reveal that the MOIPLE distribution can model non-monotonic hazard behaviour. Specifically, the hazard initially increases, reaches a maximum, and subsequently decreases. Such behaviour is commonly referred to as a unimodal or upside-down bathtub hazard rate. This behaviour implies that the probability of failure does not remain constant throughout the lifetime of the system. Instead, the risk gradually builds up during the early stages, attains a peak, and then declines afterwards.

The quantile function is the inverse of the cdf.

$$Q(u) = F^{-1}(u), \tag{5}$$

where $u \sim \text{unif}(0,1)$ (Nelsen, 2006; Ross, 2014).

The quantile function of X has the MOIPLE distribution; for $p \in (0,1)$, it is obtained by inverting the CDF (Sadiq *et al*. (2022); Sadiq *et al*. (2023a); Sadiq *et al*. (2023b); Sadiq *et al*. (2022c); Sadiq *et al*. (2024); Sadiq *et al*. (2025)) presented in equation (2) as follows:

$$p = \frac{1}{1+\theta\left(e^{\lambda x^{-\beta}}-1\right)^{\alpha}},$$

by simplifying, we obtain the Quantile Function as:

$$Q(p) = x = \left(\frac{\lambda}{\ln\left[1 + \left(\frac{1-p}{\theta p}\right)^{\frac{1}{\alpha}}\right]} \right)^{\frac{1}{\beta}} \tag{6}$$

3. Maximum Likelihood Estimation (MLE)

In many survival studies, the exact failure time is not observed for every subject due to loss to follow-up, withdrawal, or the study ending before failure occurs. Such observations are recorded as censored observations. For each unit, the observed data are (x_i, δ_i), where $x_i = \min(T_i, C_i)$ is the observed time, T_i is the true failure time, C_i is the censored time, and $\delta_i = 0$ indicates censoring while $\delta_i = 1$ indicates an observed failure, for $i = 1, \ldots, n$.

Let $d = \sum_{i=1}^{n} \delta_i$ (number of failures), then

$$L(x_i, \delta_i) = \prod_{i=1}^{n} \left[f(x_i) \right]^{\delta_i} \left[S(x_i) \right]^{1-\delta_i} \tag{7}$$

Substituting Equation (1) and Equation (3) into Equation (7) gives:

$$L(x_i, \delta_i) = \prod_{i=1}^{n} \left[\frac{\theta\alpha\beta\lambda x_i^{-(\beta+1)} e^{\lambda x_i^{-\beta}} \left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha-1}}{\left[1 + \theta\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha}\right]^2} \right]^{\delta_i} \times \prod_{i=1}^{n} \left[\frac{\theta\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha}}{1 + \theta\left(e^{-\lambda x_i^{-\beta}} - 1\right)^{\alpha}} \right]^{1-\delta_i} \tag{8}$$

Taking the logarithm of (8) and expanding it, we have:

$$\begin{aligned} \ell &= \sum_{i=1}^{n} \delta_i \ln\theta + \sum_{i=1}^{n} \delta_i \ln\alpha + \sum_{i=1}^{n} \delta_i \ln\beta + \sum_{i=1}^{n} \delta_i \ln\lambda - (\beta+1)\sum_{i=1}^{n} \delta_i \ln x_i + \lambda \sum_{i=1}^{n} \delta_i \ln x_i^{-\beta} \\ &+ (\alpha-1)\sum_{i=1}^{n} \delta_i \ln\left(e^{\lambda x_i^{-\beta}} - 1\right) - 2\sum_{i=1}^{n} \delta_i \ln\left[1 + \theta\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha}\right] + \sum_{i=1}^{n} (1-\delta_i) \ln\theta \\ &+ \alpha \sum_{i=1}^{n} (1-\delta_i) \ln\left(e^{\lambda x_i^{-\beta}} - 1\right) - \sum_{i=1}^{n} (1-\delta_i) \ln\left[1 + \theta\left(e^{\lambda x_i^{-\beta}} - 1\right)^{\alpha}\right] \end{aligned}$$

Since $\sum_{i=1}^{n} \delta_i + \sum_{i=1}^{n} (1-\delta_i) = n$

The log-likelihood simplifies to

$$\ell = n\ln\theta + d\ln\alpha + d\ln\beta + d\ln\lambda - (\beta+1)\sum_{i=1}^{n}\delta_i \ln x_i + \lambda\sum_{i=1}^{n}\delta_i \ln x_i^{-\beta}$$
$$+(\alpha-1)\sum_{i=1}^{n}\delta_i \ln\left(e^{\lambda x_i^{-\beta}}-1\right) + \alpha\sum_{i=1}^{n}(1-\delta_i)\ln\left(e^{\lambda x_i^{-\beta}}-1\right) - 2\sum_{i=1}^{n}\delta_i \ln\left[1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\right] \quad (9)$$
$$-\sum_{i=1}^{n}(1-\delta_i)\ln\left[1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\right]$$

To derive the score function, we differentiate (9) partially with respect to each parameter and equate to zero; the derivatives are given as follows:

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n}\delta_i \frac{\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}} - \sum_{i=1}^{n}(1-\delta_i)\frac{\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}} = 0 \quad (10)$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{d}{\alpha} + \sum_{i=1}^{n}\delta_i \ln\left(e^{\lambda x_i^{-\beta}}-1\right) + \sum_{i=1}^{n}(1-\delta_i)\ln\left(e^{\lambda x_i^{-\beta}}-1\right)$$
$$-2\sum_{i=1}^{n}\delta_i \frac{\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\ln\left(e^{\lambda x_i^{-\beta}}-1\right)}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}} - \sum_{i=1}^{n}(1-\delta_i)\frac{\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\ln\left(e^{\lambda x_i^{-\beta}}-1\right)}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}} = 0 \quad (11)$$

$$\frac{\partial\ell}{\partial\beta} = \frac{d}{\beta} - \sum_{i=1}^{n}\delta_i \ln x_i - \lambda\sum_{i=1}^{n}\delta_i x_i^{-\beta}\ln x_i - (\alpha-1)\sum_{i=1}^{n}\delta_i \frac{\lambda x_i^{-\beta}\ln x_i e^{\lambda x_i^{-\beta}}}{e^{\lambda x_i^{-\beta}}-1}$$
$$-\alpha\sum_{i=1}^{n}(1-\delta_i)\frac{\lambda x_i^{-\beta}\ln x_i e^{\lambda x_i^{-\beta}}}{e^{\lambda x_i^{-\beta}}-1} + 2\theta\alpha\sum_{i=1}^{n}\delta_i \frac{\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha-1}\lambda x_i^{-\beta}\ln x_i e^{\lambda x_i^{-\beta}}}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}} \quad (12)$$
$$-\theta\alpha\sum_{i=1}^{n}(1-\delta_i)\frac{\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha-1}\lambda x_i^{-\beta}\ln x_i e^{\lambda x_i^{-\beta}}}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}} = 0$$

and

$$\frac{\partial\ell}{\partial\lambda} = \frac{d}{\lambda} + \sum_{i=1}^{n}\delta_i x_i^{-\beta} + (\alpha-1)\sum_{i=1}^{n}\delta_i \frac{x_i^{-\beta}e^{\lambda x_i^{-\beta}}}{e^{\lambda x_i^{-\beta}}-1} + \alpha\sum_{i=1}^{n}(1-\delta_i)\frac{x_i^{-\beta}e^{\lambda x_i^{-\beta}}}{e^{\lambda x_i^{-\beta}}-1}$$
$$-2\theta\alpha\sum_{i=1}^{n}\delta_i \frac{\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha-1}x_i^{-\beta}e^{\lambda x_i^{-\beta}}}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}} - \sum_{i=1}^{n}(1-\delta_i)\frac{\theta\alpha\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha-1}x_i^{-\beta}e^{\lambda x_i^{-\beta}}}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}} = 0 \quad (13)$$

Equations (10) to (13) are non-linear. Hence, the MLEs of the model parameters are the solution of these non-linear equations in (10) to (13). These equations are very difficult to obtain, so L-BFGS-B iterative procedures are employed.

The precision of the maximum likelihood estimators is assessed by the observed Fisher Information Matrix. Consisting of the second-order partial derivatives of the log-likelihood function evaluated at the maximum likelihood estimates. The inverse of the observed Fisher Information Matrix provides the estimated variance-covariance matrix, from which the standard errors and approximate confidence intervals of the parameter estimates are computed. For the MOIPLE distribution with parameter vector, $\Theta=(\alpha,\beta,\lambda,\theta)$. The Fisher Information Matrix is a 4×4 matrix.

$$I(\Theta)=-\begin{pmatrix} \ell_{\alpha\alpha} & \ell_{\alpha\beta} & \ell_{\alpha\lambda} & \ell_{\alpha\theta} \\ \ell_{\beta\alpha} & \ell_{\beta\beta} & \ell_{\beta\lambda} & \ell_{\beta\theta} \\ \ell_{\lambda\alpha} & \ell_{\lambda\beta} & \ell_{\lambda\lambda} & \ell_{\lambda\theta} \\ \ell_{\theta\alpha} & \ell_{\theta\beta} & \ell_{\theta\lambda} & \ell_{\theta\theta} \end{pmatrix}$$

where the diagonal elements ($\frac{\partial^2\ell}{\partial\alpha^2},\frac{\partial^2\ell}{\partial\beta^2},\frac{\partial^2\ell}{\partial\lambda^2}$ and $\frac{\partial^2\ell}{\partial\theta^2}$) of $I(\Theta)$ are the variances of the parameters α,β,λ and θ , respectively, while the off-diagonal elements are the covariances. The diagonal elements of $I(\Theta)$ are:

$$\frac{\partial^2\ell}{\partial\alpha^2}=-\frac{d}{\alpha^2}-2\sum_{i=1}^{n}\delta_i\frac{\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\left[\ln\left(e^{\lambda x_i^{-\beta}}-1\right)\right]^2}{\left[1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\right]^2}+\sum_{i=1}^{n}(1-\delta_i)\frac{\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\left[\ln\left(e^{\lambda x_i^{-\beta}}-1\right)\right]^2}{\left[1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\right]^2}=0$$

$$\frac{\partial^2\ell}{\partial\beta^2}=-\frac{d}{\beta^2}+\lambda\sum_{i=1}^{n}\delta_i x_i^{-\beta}(\ln x_i)^2+(\alpha+1)\sum_{i=1}^{n}\delta_i\frac{\lambda x_i^{-\beta}(\ln x_i)^2 e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1-\lambda x_i^{-\beta}\right)}{\left(e^{\lambda x_i^{-\beta}}-1\right)^2}$$

$$+\alpha\sum_{i=1}^{n}(1-\delta_i)\frac{\lambda x_i^{-\beta}(\ln x_i)^2 e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1-\lambda x_i^{-\beta}\right)}{\left(e^{\lambda x_i^{-\beta}}-1\right)^2}-2\sum_{i=1}^{n}\delta_i\theta\alpha\lambda x_i^{-\beta}(\ln x_i)^2 e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha-2}$$

$$\left[\frac{\left(e^{\lambda x_i^{-\beta}}-1\right)\left(1-\lambda x_i^{-\beta}\right)-\lambda x_i^{-\beta}(\alpha-1)e^{\lambda x_i^{-\beta}}}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}+\frac{\theta\alpha\lambda x_i^{-\beta}e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}{\left[1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\right]^2}\right]-$$

$$\sum_{i=1}^{n}(1-\delta_i)\theta\alpha\lambda x_i^{-\beta}(\ln x_i)^2 e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha-2}$$

$$\left[\frac{\left(e^{\lambda x_i^{-\beta}}-1\right)\left(1-\lambda x_i^{-\beta}\right)-\lambda x_i^{-\beta}(\alpha-1)e^{\lambda x_i^{-\beta}}}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}+\frac{\theta\alpha\lambda x_i^{-\beta}e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}{\left[1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\right]^2}\right]=0$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=-\frac{d}{\lambda^2}-(\alpha+1)\sum_{i=1}^{n}\delta_i\frac{\lambda x_i^{-2\beta}e^{\lambda x_i^{-\beta}}}{\left(e^{\lambda x_i^{-\beta}}-1\right)^2}-\alpha\sum_{i=1}^{n}(1-\delta_i)\frac{x_i^{-2\beta}e^{\lambda x_i^{-\beta}}}{\left(e^{\lambda x_i^{-\beta}}-1\right)^2}-2\sum_{i=1}^{n}\delta_i\theta\alpha x_i^{-2\beta}e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha-2}$$

$$\left[\frac{(\alpha-1)e^{\lambda x_i^{-\beta}}+1}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}+\frac{\theta\alpha e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}{\left[1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\right]^2}\right]-\sum_{i=1}^{n}(1-\delta_i)\theta\alpha x_i^{-2\beta}e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha-2}$$

$$\left[\frac{(\alpha-1)e^{\lambda x_i^{-\beta}}+1}{1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}+\frac{\theta\alpha e^{\lambda x_i^{-\beta}}\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}}{\left[1+\theta\left(e^{\lambda x_i^{-\beta}}-1\right)^{\alpha}\right]^2}\right]=0$$

It is important to note that the 4×4 observed information matrix $I=I(\Theta)$ (for $\Theta=\alpha,\beta,\lambda,\theta$) is multivariate normal $N_4\left(0,I^{-1}\left(\hat{\Theta}\right)\right)$. The multivariate normal $N_4\left(0,I^{-1}\left(\hat{\Theta}\right)\right)$ distribution is used to construct approximate CIs for the parameters. The approximate 100(1-γ)% CIs for v is given by:

$$\hat{v}\pm Z_{\frac{\gamma}{2}}SE(\hat{v}),$$

where $Z_{\frac{\gamma}{2}}$ is the $\left[100(1-\gamma)/2\right]^{th}$ standard normal percentile and $SE(\hat{v})$ is the standard deviation for the ML estimators.

4. Results

In this section, a simulation study was conducted to ascertain the performance of the MLEs. In addition, a real-life data application was also provided.

4.1 Simulation Study

The simulation study is carried out to evaluate the performance of the MLE estimators. Measures like Root Mean Square Errors (RMSEs), standard errors (SEs) and Bias were used in assessing the performance of the MLEs. The simulation study is described as follows:

a. Random samples of size *n* = 30, 50, 75, 100 and 200 are generated from the MOIPLE distribution.
b. A sets of parameter values are chosen as; α = 0.8, β = 2.5, λ = 1.8 and θ = 2.3.
c. Estimates of the parameters are obtained
d. These procedures are replicated 1000 times

Table 1 shows the estimates of the maximum likelihood estimators together with their SEs, bias and RMSEs. The results demonstrate that the performance of the estimators improves as the sample size increases from 30 to 200. Across all four parameters, the estimated values move progressively closer to their corresponding true parameter values with increasing sample size, indicating consistency of the maximum likelihood estimators. In addition, the standard errors decrease steadily, reflecting improved precision of the estimates. The absolute bias also becomes smaller for most parameters, particularly when the sample size reaches 100 and 200, while the RMSE values consistently decline with increasing sample size. These findings suggest that the maximum likelihood method provides reliable and asymptotically efficient parameter estimates for the MOIPLE distribution under right-censored data.

Table 1: Bias, SEs and RMSEs of the MLE for the MOIPLE distribution.

N	Parameter	Estimate	SE	Bias	RMSE
30	α	1.79973	0.01427	0.99973	0.30977
	β	2.64119	0.03456	0.14119	0.76322
	λ	2.14188	0.02817	0.34188	0.70053
	θ	2.99167	0.02884	0.69167	0.69774
50	α	1.78057	0.00736	0.98057	0.18003
	β	2.60348	0.02436	0.10348	0.60122
	λ	2.0913	0.02183	0.2913	0.60546
	θ	2.75708	0.02278	0.45708	0.60475
75	α	0.89912	0.00609	0.09912	0.16317
	β	2.53015	0.01743	0.03015	0.46766
	λ	1.97417	0.01647	0.17417	0.47419
	θ	2.71684	0.01803	0.41684	0.5163
100	α	0.80854	0.00398	0.00854	0.11337
	β	2.5051	0.01262	0.0051	0.35958
	λ	1.9382	0.01327	0.1382	0.40253
	θ	2.35326	0.01552	0.05326	0.46587
200	α	0.79896	0.00267	-0.001	0.08205
	β	2.47636	0.00808	-0.0236	0.24809
	λ	1.85712	0.0081	0.05712	0.25426
	θ	2.32935	0.01121	0.02935	0.34996

4.2 Real-life data application

This subsection evaluates the practical applicability of the MOIPLE model using a real medical dataset involving right-censored survival times. The MOIPLE model is assessed under maximum likelihood estimation and compared with some competing models, including Marshall-Olkin Inverse Power Lomax (MOIPL) developed by Hassan and Nassr (2021) and Marshall-Olkin Logistic-Exponential (MOLE) model developed by Mansoor *et al*. (2019). Model selection criteria such as AIC, BIC and CAIC were used to determine the best-fitted model.

The data used in this study consist of survival times of patients with acute lymphoblastic leukaemia following bone marrow transplantation and were used by Usman *et al.* (2021). It comprises 90 patients, of whom 46 received allogeneic bone marrow transplantation, and 44 received autologous transplantation. During the follow-up period, 69 patients experienced recurrence of leukaemia, while the remaining 21 observations were right-censored, indicating that no recurrence was observed before the end of the study. Consequently, this dataset is appropriate for evaluating lifetime models designed for right-censored survival data.

Table 2 presents the maximum likelihood estimates and goodness-of-fit statistics for the Leukaemia data using the MOIPLE model and two other distributions: MOIPL and MOLE. The results indicate that the MOIPLE model provides the best fit to the data, having the highest log-likelihood value (-64.87067), which implies that it explains the observed data better than the other models. Furthermore, the MOIPLE model yields the smallest values of the AIC (137.7413), BIC (147.7406) and CAIC (151.7406).

Table 2: Maximum likelihood estimates of Leukaemia data.

Models	Parameters	Estimate	Goodness of Fit Measures	Rank
MOIPLE	α	0.097598	LL (-64.87067)	1
	β	0.519944	AIC (137.7413)	
	λ	13.574097	BIC (147.7406)	
	θ	0.141732	CAIC (151.7406)	
MOIPL	α	0.378999	LL (-67.46317)	2
	β	146.293926	AIC (142.9263)	
	λ	72.296351	BIC (152.9256)	
	θ	0.082531	CAIC (156.9256)	
MOLE	α	0.979148	LL (-77.10587)	3
	β	0.003914	AIC (160.2117)	
	λ	0.002700	BIC (167.7112)	
			CAIC (170.7112)	

The MOIPL model ranks second, with a log-likelihood of -67.46317 and slightly larger values of AIC (142.9263), BIC (152.9256), and CAIC (156.9256). Although it provides a satisfactory fit, its performance is inferior to that of the MOIPLE model. Among all the fitted models, the MOLE model performs the poorest, having the lowest log-likelihood (-77.10587) and the largest values of AIC (160.2117), BIC (167.7112), and CAIC (170.7112). These results indicate that it provides the least adequate fit to the Leukaemia data. Since lower values of these information criteria indicate a better balance between model fit and complexity, the MOIPLE distribution is preferred over the MOIPL and MOLE models.

6. Conclusion

This study investigated the maximum likelihood estimation of the parameters of the MOIPLE distribution under right-censoring. The quantile function and maximum likelihood estimation procedure were successfully derived. In addition, the observed Fisher Information Matrix was formulated to facilitate interval estimation through the estimation of standard errors and approximate confidence intervals.

Monte Carlo simulation study involving 1,000 replications and sample sizes of 30, 50, 75, 100, and 200 was conducted to assess the finite-sample performance of the maximum likelihood estimators.

The simulation results demonstrated that the estimators are consistent, as the parameter estimates converge towards their true values with increasing sample size. Furthermore, the standard errors, bias, and root mean square errors generally decrease as the sample size increases, indicating improved estimation accuracy and precision.

The practical usefulness of the model was illustrated using a right-censored leukaemia dataset. The MOIPLE distribution was compared with the MOIPL and MOLE distributions using log-likelihood, AIC, BIC and CAIC. The results showed that the MOIPLE distribution achieved the highest log-likelihood and the lowest AIC, BIC, and CAIC values among the competing models, indicating superior goodness-of-fit for the analysed dataset.

The findings suggest that the MOIPLE distribution is a flexible and effective lifetime model for analysing right-censored survival data. The combination of favourable theoretical properties, satisfactory estimation performance and superior empirical fit demonstrates its potential usefulness in reliability analysis, medical survival studies, and other applications involving censored lifetime data. Future research may consider Bayesian estimation, progressive censoring schemes and applications of the MOIPLE distribution to other real-life datasets to further evaluate its performance and extend its applicability.

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Appendix

Table 1: Survival times of a sample of 90 patients, of whom 46 received allogeneic bone marrow transplantation (TRT, 0) and 44 received autologous transplantation (TRT, 1) for leukaemia recurrence. During the follow-up period, 69 patients experienced the event (Status, 1), while the remaining 21 observations were right-censored (Status, 0).

t	TRT	Status	t	TRT	Status	t	TRT	Status
0.030137	0	1	5	0	0	4.690411	0	0
0.038356	0	1	0.057534	1	1	4.780822	0	0
0.063014	0	1	0.109589	1	1	4.986301	0	0
0.084932	0	1	0.115068	1	1	4.041096	1	0
0.087671	0	1	0.136986	1	1	4.205479	1	0
0.09589	0	1	0.145205	1	1	4.205479	1	0
0.139726	0	1	0.147945	1	1	2.243836	0	1
0.161644	0	1	0.153425	1	1	2.506849	0	0
0.169863	0	1	0.167123	1	1	2.646575	0	0
0.213699	0	1	0.175342	1	1	3.038356	0	0
0.213699	0	1	0.183562	1	1	3.172603	0	0
0.216438	0	1	0.2	1	1	3.441096	0	1
0.238356	0	1	0.208219	1	1	4.421918	0	0
0.271233	0	1	0.216438	1	1	4.435616	0	0
0.273973	0	1	0.221918	1	1	4.586301	0	0
0.386301	0	1	0.241096	1	1	0.610959	1	1
0.438356	0	1	0.260274	1	1	0.613699	1	1
0.454795	0	1	0.268493	1	1	0.758904	1	1
0.591781	0	1	0.268493	1	1	1.983562	1	0
0.6	0	1	0.271233	1	1	1.99726	1	0
0.643836	0	1	0.284932	1	1	2.010959	1	1
0.684932	0	1	0.287671	1	1	2.884932	1	0
0.739726	0	1	0.290411	1	1	2.99726	1	0
0.857534	0	1	0.306849	1	1	3.265753	1	0
0.909589	0	1	0.358904	1	1			
0.964384	0	1	0.40274	1	1			
1.008219	0	1	0.468493	1	1			
1.282192	0	1	0.471233	1	1			
1.345205	0	1	0.490411	1	1			
1.4	0	1	0.517808	1	1			
1.526027	0	1	0.534247	1	1			
1.720548	0	0	0.545205	1	1			
1.989041	0	0	0.583562	1	1			