On the Sine-Weibull-G Family of Distributions: Statistical Properties and Application

Isa Abubakar Ibrahim[1*], Abubakar Yahaya[2], Abubakar Usman[2], Jamila Abdullahi[2], Jamilu Yunusa Falgore[2], Dewu Mustapha Muhammad[2] and Saudat Ali Adamu[2]

[1]Department of Mathematics and Statistics, Federal University, Kashere, Gombe, Nigeria.
[2]Department of Statistics, Ahmadu Bello University, Zaria, Nigeria.
*Correspondence: isaibrahim951@yahoo.com

Abstract
Lifetime distributions are essential tools for modelling continuous data arising in engineering, physics, medicine and reliability studies. However, many classical lifetime models lack the flexibility needed to adequately capture diverse distributional shapes and hazard rate behaviours observed in practice. Motivated by this limitation, this paper proposes a new class of continuous probability distributions called the Sine-Weibull-G (SW-G) family by incorporating the Weibull-transformed odds ratio into the Sine generator. The aim is to develop the statistical properties of the proposed family and assess its performance in practical applications. Several important statistical properties of the proposed family are derived, including the cumulative distribution function, probability density function, quantile function, moments, moment generating function, Rényi entropy and order statistics. A special case, the Sine-Weibull-Exponential (SW-E) distribution, is obtained by adopting the Exponential distribution as the baseline model. Parameters are estimated using the maximum likelihood method, while Monte Carlo simulation is employed to evaluate the consistency and finite-sample performance of the estimators. The proposed model is applied to two real-life datasets. For the first dataset, the SW-E model achieves the highest log-likelihood (196.595) and the lowest AIC (-387.191), BIC (-377.748), and CAIC (-374.748). Similarly, for the second dataset, it attains the highest log-likelihood (-14.1001) and the smallest AIC (34.2002), BIC (40.6296) and CAIC (34.6070), outperforming all competing models. These results established the SW-G family as a robust and flexible framework for modelling continuous lifetime data across diverse scientific applications.

Keywords: Sine-Weibull-G family, Exponential distribution, Sine-Weibull-Exponential distribution, Maximum likelihood estimation, Monte Carlo simulation, Goodness-of-fit.

1. Introduction

Statistical distributions form the backbone of probabilistic modelling across engineering, medical, actuarial, environmental and economic sciences. Classical distributions such as the exponential, Weibull, gamma and log-normal have historically provided reasonably good fits to lifetime and reliability data. However, most real-world data sets exhibit skewness, multimodality and hazard shapes (bathtub, upside-down bathtub, monotone increasing/decreasing) that classical models cannot always accommodate. This limitation has motivated statisticians to develop more flexible extensions of existing baseline distributions by adding one or more shape parameters, thereby producing families of distributions that are richer in flexibility without unduly complicating parameter estimation. One influential strategy is the transformed-transformer (T-X) approach introduced by Alzaatreh *et al.* (2013), which allows any baseline cumulative distribution function to be transformed through a generator random variable to produce a new, more flexible family. Building on this idea, several generator families have been proposed, including the Beta-G family by Eugene *et al.* (2002), the Kumaraswamy-G family by Cordeiro & de Castro (2011), the transmuted Exponentiated generalized-G family by Yousof *et al.* (2017)

and the Weibull-G family by Bourguignon *et al.* (2014). The Weibull-G family is particularly attractive because it uses the Weibull distribution, already central to reliability theory as a generator applied to any baseline distribution, producing a new family whose density can take increasing, decreasing, bathtub, and reversed-J hazard shapes while adding relatively few extra parameters. Because of this balance between flexibility and parsimony, the Weibull-G family has since been used as the baseline for numerous further extensions, including the beta Weibull-G family and various odd-generalised and transmuted Weibull-G families by Alizadeh *et al.* (2018).

A separate and more recent line of research introduces trigonometric functions as generators. Kumar *et al.* (2015) proposed a one-parameter free sine transformation applied to the cdf of a baseline distribution, motivated by the periodicity, boundedness and simple analytic tractability of trigonometric functions. This idea was formalised into a general class by Souza *et al.* (2019a), who introduced the Sine-G family of distributions by applying a sine transformation directly to the cdf of any baseline distribution. The appeal of the Sine-G construction lies in the fact that it does not introduce any additional parameter to the baseline model, yet it can noticeably improve tail behaviour and goodness of fit relative to the baseline distribution alone.

Motivated by the success of the Sine-G family, numerous extensions have since been developed by researchers to further increase its flexibility. Souza *et al.* (2019b) proposed the closely related Cos-G family, based on a cosine rather than sine transformation, while Souza *et al.* (2023) introduced the Tan-G family using the tangent function. Chesneau & Jamal (2020) extended the sine transformation by compounding it with the Kumaraswamy-G family to obtain the sine Kumaraswamy-G family, adding extra shape parameters for greater flexibility, with a detailed special case built on the exponential baseline. Al-Babtain *et al.* (2020) proposed the sine Topp-Leone-G family, combining the sine transformation with the two-parameter Topp-Leone-G generator to better capture bathtub-shaped hazard rates. Muhammad *et al.* (2021) introduced the Exponentiated sine-G family by raising the Sin-G cdf to a positive power, thereby adding a single extra shape parameter while preserving the boundedness of the trigonometric transformation. Jamal *et al.* (2021) proposed a further generalisation, the transformed Sine-G family, which nests the original Sine-G family as a special case and allows additional control over skewness and kurtosis. More recently, Rajkumar & Sakthivel (2022) combined the Marshall-Olkin mechanism with the sine transformation to develop the Marshall-Olkin sine-G family for survival analysis applications. Collectively, these extensions confirm that the sine transformation is a flexible and easily generalizable building block that can be compounded with virtually any baseline or generator family, which further supports the plausibility and relevance of combining it with the Weibull-G family, as proposed in this paper.

Despite the extensive development of both polynomial/exponential type of generator families such as the Weibull-G family, and trigonometric generator families such as the Sine-G class and its extensions, comparatively little attention has been paid to combining the two mechanisms, that is, using an already generalised family such as the Weibull-G family as the baseline distribution inside a trigonometric sine transformation. Such a combination is of interest for at least three reasons. First, the Weibull-G family already provides considerable flexibility in hazard shape through its own shape parameter and wrapping it inside the bounded, monotone sine transformation can further sharpen tail and shape behaviour without the parameter proliferation typical of compounding two shape parameter families directly. Second, because the sine transformation contributes no new parameter, the resulting Sine-Weibull-G family remains as parsimonious as the underlying Weibull-G family, which is desirable for maximum likelihood estimation with small to moderate sample sizes. Third, choosing the exponential distribution as the baseline inside the Weibull-G generator yields a fully closed-form, three-parameter special model here called the Sine-Weibull-Exponential (SW-E) distribution with closed-form survival, hazard, and quantile functions, making it computationally convenient for both theoretical derivation and simulation-based inference.

This paper aims to introduce and comprehensively study a new generator, the Sine-Weibull-G (SW-G) family of distributions, obtained by embedding the Weibull-G family of Bourguignon *et al.* (2014) as the baseline distribution within the Sine-G transformation of Kumar *et al.* (2015), and to examine in detail its special case constructed from the exponential baseline. To achieve this aim, the specific objectives of the paper are to: (i) derive the general cumulative distribution and probability density functions of the proposed Sine-Weibull-G family for an arbitrary baseline distribution; (ii) establish key structural properties of the family, including the survival function, hazard rate function, quantile function and the density of order statistics; (iii) specialize the general family to the case where the baseline distribution is exponential, yielding the new three parameter Sine-Weibull-Exponential (SW-E) distribution, and examine its density and hazard shapes; (iv) estimate the unknown parameters of the SW-G distribution using the method of maximum likelihood, and conduct a Monte Carlo simulation study across increasing sample sizes to assess the consistency, bias, and root mean square error of the resulting estimators; (v) demonstrate the practical usefulness and comparative fit of the SW-E distribution relative to competing lifetime models using real data application(s).

2. Materials and Methods

2.1. The Sine-G Family of Distributions

A random variable X is said to follow the Sine-G family of distributions according to Kumar *et al.* (2015) if its CDF and PDF are respectively given as;

$$F(x;\zeta)=\int_0^{\frac{\pi}{2}H(x;\zeta)}\cos t\,dt=\sin\left(\frac{\pi}{2}H(x;\zeta)\right), \tag{1}$$

and

$$f(x;\zeta)=\frac{\pi}{2}h(x;\zeta)\cos\left(\frac{\pi}{2}H(x;\zeta)\right), \tag{2}$$

where $H(x;\zeta)$ and $h(x;\zeta)$ are the CDF and PDF of any baseline distribution with parameter vector ζ.

2.2. The Weibull-G Family of Distributions

Bourguignon *et al.* (2014) defined the Weibull-G family of distributions. A random variable x is said to follow the Weibull-G family of distributions if its CDF and PDF are respectively given as;

$$H\left(x;\alpha,\beta,\zeta\right)=\int_0^{\frac{G(x;\zeta)}{1-G(x;\zeta)}}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx=1-\exp\left\{-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right\}. \tag{3}$$

$$x>0,\alpha>0,\beta>0$$

and

$$h(x;\alpha,\beta,\zeta)=\alpha\beta g(x;\zeta)\frac{G(x;\zeta)^{\beta-1}}{\bar{G}(x;\zeta)^{\beta+1}}\exp\left\{-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right\}. \tag{4}$$

$$x>0,\alpha>0,\beta>0$$

where $G(x;\zeta)$ and $g(x;\zeta)$ are the CDF and PDF of any baseline distribution with parameter vector ζ. Also, α and β are the scale and shape parameters, respectively.

2.3. The Proposed Sine-Weibull-G Family of Distributions

The CDF of the new Sine-Weibull-G Family is given by:

$$F(x;\alpha,\beta,\zeta)=\sin\left(\frac{\pi}{2}\left[1-\exp\left\{-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right\}\right]\right). \tag{5}$$

with the corresponding pdf given by:

$$f(x;\alpha,\beta,\zeta)=\frac{\pi}{2}\alpha\beta\frac{g(x;\zeta)G(x;\zeta)^{\beta-1}}{\bar{G}(x;\zeta)^{\beta+1}}\exp\left\{-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right\}\times\cos\left(\frac{\pi}{2}\left[1-\exp\left\{-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right\}\right]\right). \tag{6}$$

where $x>0$, $\alpha>0$, and $\beta>0$. The parameters α and β denote the scale and shape parameters, respectively, whereas ζ denotes the parameter vector of the baseline distribution.
The survival function $S(x)$, hazard function $h(x)$, reversed hazard function $r(x)$ and the quantile functions $Q(x)$ of the SW-G are presented in equations (7) to (10).

$$S(x;\alpha,\beta,\zeta)=1-\sin\left(\frac{\pi}{2}\left[1-\exp\left\{-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right\}\right]\right). \tag{7}$$

$$h(x;\alpha,\beta,\zeta)=\frac{\frac{\pi}{2}\alpha\beta\frac{g(x;\zeta)G(x;\zeta)^{\beta-1}}{\bar{G}(x;\zeta)^{\beta+1}}\exp\left\{-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right\}\cos\left(\frac{\pi}{2}\left[1-\exp\{-\alpha(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)})^{\beta}\}\right]\right)}{1-\sin\left(\frac{\pi}{2}\left[1-\exp\{-\alpha(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)})^{\beta}\}\right]\right)}. \tag{8}$$

$$r(x;\alpha,\beta,\zeta)=\frac{\pi}{2}\alpha\beta\frac{g(x;\zeta)G(x;\zeta)^{\beta-1}}{\bar{G}(x;\zeta)^{\beta+1}}\exp\left[-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right]\cot\left[\frac{\pi}{2}\left(1-\exp\left\{-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right\}\right)\right]. \tag{9}$$

$$Q(u;\alpha,\beta,\zeta)=G^{-1}\left(\frac{\left[-\frac{1}{\alpha}\ln\left(1-\frac{2}{\pi}\arcsin(u)\right)\right]^{1/\beta}}{1+\left[-\frac{1}{\alpha}\ln\left(1-\frac{2}{\pi}\arcsin(u)\right)\right]^{1/\beta}}\right),\quad 0<u<1, \tag{10}$$

where $G^{-1}(\cdot)$ is the quantile function of the baseline distribution with CDF $G(x;\zeta)$.

2.4. Statistical Properties

This section presents some statistical properties of the SW-G family of distributions.

2.4.1. Moments

The moments are fundamental for studying various distributional characteristics, including measures of central tendency (mean, $\mu_{1'}$), dispersion (variance, $\mu_{2'}-(\mu_{1'})^2$, skewness, and kurtosis of the SW-G family. The r^{th} raw moment of a continuous random variable X with PDF $f(x;\alpha,\beta,\zeta)$ is defined as

$$\mu_{r'}=E[X^r]=\int_{-\infty}^{\infty}x^r f(x;\alpha,\beta,\zeta)dx$$

Therefore, the moment of the Sine-Weibull-G is obtained as follows:

$$\mu_{r'}=\frac{\pi}{2}\alpha\sum_{k=0}^{\infty}\sum_{j=0}^{2k}\frac{(-1)^{k+j}}{(2k)!}\binom{2k}{j}\left(\frac{\pi}{2}\right)^{2k}\int_0^{\infty}\left[G^{-1}\left(\frac{u^{1/\beta}}{1+u^{1/\beta}}\right)\right]^r e^{-\alpha(j+1)u},du, \tag{11}$$

2.4.2. Moment Generating Function

The moment generating function of a continuous random variable X, when it exists, is defined as

$$M_X(t)=\mathbb{E}(e^{tX})=\int_{-\infty}^{\infty}e^{tx}f(x;\alpha,\beta,\zeta),dx, \tag{12}$$

for all t in some open interval containing zero. The moment generating function of the Sine-Weibull-G family of Distributions is given by:

$$M_X(t)=\frac{\pi}{2}\alpha\sum_{k=0}^{\infty}\sum_{j=0}^{2k}\frac{(-1)^{k+j}}{(2k)!}\binom{2k}{j}\left(\frac{\pi}{2}\right)^{2k}\int_0^{\infty}\exp\left[t,G^{-1}\left(\frac{v^{1/\beta}}{1+v^{1/\beta}}\right)\right]e^{-\alpha(j+1)v},dv, \tag{13}$$

2.4.2. Entropy

The Rényi entropy of order δ, introduced by Rényi (1961), is a generalisation of the Shannon entropy and quantifies the information or uncertainty contained in a random observation. For a continuous random variable X with probability density function $f(x)$, the Rényi entropy is defined as

$$I_R(\delta)=\frac{1}{1-\delta}\log\left(\int_{-\infty}^{\infty}f(x)^{\delta},dx\right),\quad \delta>0,\quad \delta\neq 1. \tag{14}$$

The entropy for the Sine-Weibull-G family of distributions is given by:

$$I_R(\delta)=\frac{1}{1-\delta}\left[\delta\log\left(\frac{\pi}{2}\right)+\delta\log\alpha+\delta\log\beta+\log J(\delta)\right], \tag{15}$$

where

$$J(\delta)=\int_0^1 g\left(G^{-1}(u)\right)^{\delta-1}u^{\delta(\beta-1)}(1-u)^{-\delta(\beta+1)}\exp\left[-\delta\alpha\left(\frac{u}{1-u}\right)^{\beta}\right]\times\cos^{\delta}\left[\frac{\pi}{2}\left(1-e^{-\alpha\left(\frac{u}{1-u}\right)^{\beta}}\right)\right],du,$$

provided that the integral converges, where $g(\cdot)$ is the baseline density function and $G^{-1}()$ is the quantile function of the baseline distribution with CDF $G(x;\zeta)$.

2.4.2. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population having a PDF $f(x)$ and CDF $F(x)$; let $X_{1:n}\leq X_{2:n}\leq\cdots\leq X_{n:n}$ denote the associated order statistics. Then, David & Nagaraja (2004); Sadiq *et al.* (2022); Sadiq *et al.* (2023a); Sadiq *et al.* (2023b); Sadiq *et al.* (2022c); Sadiq *et al.* (2024); Sadiq *et al.* (2025) defined the PDF of the r^{th} order statistic, for $r=1,2,\ldots,n$, as

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}, \tag{16}$$

The order statistics of the Sine-Weibull-G are given by:

$$\begin{aligned}f_{r:n}(x)=&\frac{n!}{(r-1)!(n-r)!}\frac{\pi}{2}\alpha\beta\frac{g(x;\zeta)G(x;\zeta)^{\beta-1}}{\bar{G}(x;\zeta)^{\beta+1}}\exp\left[-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right]\times\left[\sin\left(\frac{\pi}{2}\left[1-\exp\left[-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right]\right]\right)\right]^{r-1}\\ &\times\left[1-\sin\left(\frac{\pi}{2}\left[1-\exp\left[-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right]\right]\right)\right]^{n-r}\times\cos\left(\frac{\pi}{2}\left[1-\exp\left[-\alpha\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)^{\beta}\right]\right]\right).\end{aligned} \tag{17}$$

2.5. Maximum Likelihood Estimation

Let $X_1,\ldots,X_n$ be a random sample of size n independently drawn from the SW-G family. Then the likelihood function is given below

$$L(\alpha,\beta,\zeta)=\prod_{i=1}^{n}f(x_i;\alpha,\beta,\zeta) \tag{18}$$

Substituting the density function into equation (18) yields

$$L=\prod_{i=1}^{n}\left[\frac{\pi}{2}\alpha\beta\frac{g(x;\zeta)G(x;\zeta)^{\beta-1}}{\bar{G}(x;\zeta)^{\beta+1}}\exp\left\{-\alpha T(x;\zeta)^{\beta}\right\}\cos\left(\frac{\pi}{2}\left[1-e^{-\alpha T(x;\zeta)^{\beta}}\right]\right)\right] \tag{19}$$

where, for simplicity, we define $T_i=\dfrac{G(x;\zeta)}{\bar{G}(x;\zeta)}$. Then the log-likelihood function is given by taking

logarithms of equation (3.54), which yields $\ell(\alpha,\beta,\zeta)=\log L(\alpha,\beta,\zeta)$.
Hence,

$$\begin{aligned}\ell &= n\log\left(\frac{\pi}{2}\right)+n\log\alpha+n\log\beta+\sum_{i=1}^{n}\log g(x;\zeta)+(\beta-1)\sum_{i=1}^{n}\log G(x;\zeta)\\ &\quad -(\beta+1)\sum_{i=1}^{n}\log\bar{G}(x;\zeta)-\alpha\sum_{i=1}^{n}T(x;\zeta)^{\beta}+\sum_{i=1}^{n}\log\cos\left(\frac{\pi}{2}\left[1-e^{-\alpha T(x;\zeta)^{\beta}}\right]\right)\end{aligned} \tag{20}$$

The score Equations are $\frac{\partial\ell}{\partial\alpha}=0,\quad \frac{\partial\ell}{\partial\beta}=0,\quad \frac{\partial\ell}{\partial\zeta}=0$

Differentiating the log-likelihood with respect to α yields

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}T(x;\zeta)^{\beta}+\sum_{i=1}^{n}\frac{\partial}{\partial\alpha}\log\cos(A_i) \tag{21}$$

where $A_i=\frac{\pi}{2}\left[1-e^{-\alpha T(x;\zeta)^{\beta}}\right]$ using, $\frac{d}{d\alpha}\log\cos(A_i)=-\tan(A_i)\frac{dA_i}{d\alpha}$ and

$$\frac{dA_i}{d\alpha}=\frac{\pi}{2}T(x;\zeta)^{\beta}e^{-\alpha T(x;\zeta)^{\beta}}$$

we obtained

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}T(x;\zeta)^{\beta}-\frac{\pi}{2}\sum_{i=1}^{n}T(x;\zeta)^{\beta}e^{-\alpha T(x;\zeta)^{\beta}}\tan(A_i) \tag{22}$$

Also, differentiating the log-likelihood with respect to β Using, $\frac{\partial T(x;\zeta)^{\beta}}{\partial\beta}=T(x;\zeta)^{\beta}\log T(x;\zeta)$, we obtained

$$\begin{aligned}\frac{\partial\ell}{\partial\beta}&=\frac{n}{\beta}+\sum_{i=1}^{n}\log G(x;\zeta)-\sum_{i=1}^{n}\log\bar{G}(x;\zeta)-\alpha\sum_{i=1}^{n}T(x;\zeta)^{\beta}\log T(x;\zeta)\\ &\quad-\frac{\pi}{2}\alpha\sum_{i=1}^{n}T(x;\zeta)^{\beta}\log T(x;\zeta)e^{-\alpha T(x;\zeta)^{\beta}}\tan(A_i)\end{aligned}$$

Hence,

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left(\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}\right)-\alpha\sum_{i=1}^{n}T(x;\zeta)^{\beta}\log T(x;\zeta)\left[1+\frac{\pi}{2}e^{-\alpha T(x;\zeta)^{\beta}}\tan(A_i)\right] \tag{23}$$

Also, differentiating the log-likelihood with respect to ζ by letting $\eta_i=\eta(x_i;\zeta)=\frac{\partial G(x;\zeta)}{\partial\zeta}$ since $T(x;\zeta)=\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}$ then $\frac{\partial T(x;\zeta)}{\partial\zeta}=\frac{\eta_i}{\bar{G}(x;\zeta)^2}$ therefore $\frac{\partial T(x;\zeta)^{\beta}}{\partial\zeta}=\beta T(x;\zeta)^{\beta-1}\frac{\eta_i}{\bar{G}(x;\zeta)^2}$
Then the score function for ζ becomes

$$\begin{aligned}\frac{\partial\ell}{\partial\zeta}&=\sum_{i=1}^{n}\frac{\partial}{\partial\zeta}\log g(x;\zeta)+(\beta-1)\sum_{i=1}^{n}\frac{\eta_i}{G(x;\zeta)}+(\beta+1)\sum_{i=1}^{n}\frac{\eta_i}{\bar{G}(x;\zeta)}\\ &\quad-\alpha\beta\sum_{i=1}^{n}T(x;\zeta)^{\beta-1}\frac{\eta_i}{\bar{G}(x;\zeta)^2}-\times\frac{\pi}{2}\alpha\beta\sum_{i=1}^{n}T(x;\zeta)^{\beta-1}\frac{\eta_i}{\bar{G}(x;\zeta)^2}e^{-\alpha T(x;\zeta)^{\beta}}\tan(A_i)\end{aligned} \tag{24}$$

Setting equations (22), (23) and (24) equal to zero gives the likelihood equation for α, β and ζ respectively.

2.6. Special Models of SW-G Family

Here, we consider one special model of the SW-G family along with the plots of its density and hazard rate function.

2.6.1 Sine-Weibull-Exponential (SW-E) Distribution

Let the baseline random variable follow the exponential distribution with parameter $\lambda > 0$. Then

$$G(x;\lambda) = 1 - e^{-\lambda x}, \quad x > 0, \lambda > 0 \tag{25}$$

and the survival function

$$\bar{G}(x;\lambda) = e^{-\lambda x}, \quad x > 0, \lambda > 0 \tag{26}$$

Hence

$$\frac{G(x;\lambda)}{\bar{G}(x;\lambda)} = \frac{1 - e^{-\lambda x}}{e^{-\lambda x}} = e^{\lambda x} - 1 \tag{27}$$

Substituting equation (27) into equation (5) gives the CDF of the new distribution called the SW-E distribution:

$$F(x;\alpha,\beta,\lambda) = \sin\left[\frac{\pi}{2}\left(1 - \exp\left\{-\alpha(e^{\lambda x} - 1)^{\beta}\right\}\right)\right], \quad x > 0, \quad \alpha > 0, \quad \beta > 0, \quad \lambda > 0 \tag{28}$$

We obtained the PDF by differentiating the CDF in equation (28) with respect to x, which is given as

$$f(x) = \frac{\pi}{2}\alpha\beta\lambda e^{\lambda x}(e^{\lambda x} - 1)^{\beta-1}\exp\left\{-\alpha(e^{\lambda x} - 1)^{\beta}\right\}\cos\left[\frac{\pi}{2}\left(1 - \exp\left\{-\alpha(e^{\lambda x} - 1)^{\beta}\right\}\right)\right] \tag{29}$$

$$\alpha > 0, \quad \beta > 0, \quad \lambda > 0$$

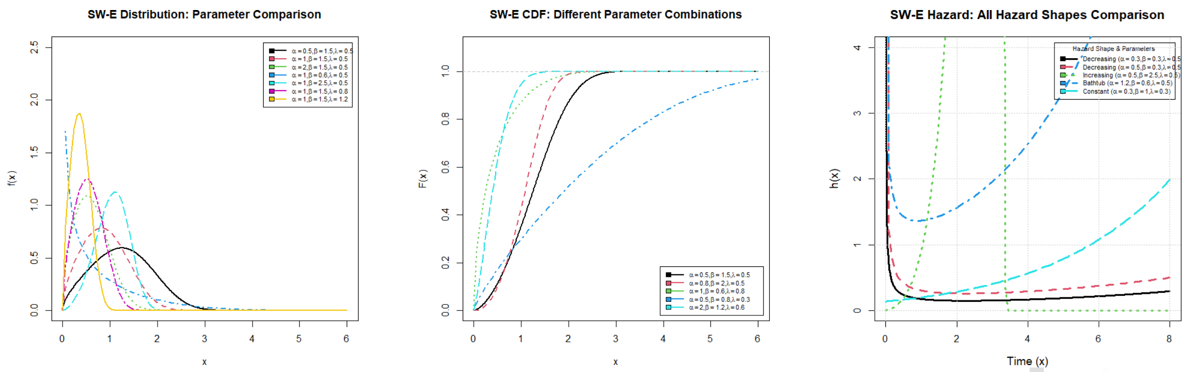
Figure 1 displays the probability density function (PDF), cumulative distribution function (CDF) and hazard rate function (HRF) of the Sine-Weibull-Exponential (SW-E) distribution under several parameter combinations of α, β, and λ, illustrating the flexibility the SW-G construction confers on its exponential special case.

Panel (a) shows that the density is capable of a variety of shapes depending on the parameter configuration, including sharply peaked and right-skewed unimodal forms, more moderately peaked unimodal forms, and near-exponential decreasing shapes. This shape flexibility indicates that the SW-E distribution is not restricted to the monotonically decreasing density inherent to the baseline exponential model, but can accommodate mode-bearing, positively skewed data patterns typical of lifetime and reliability datasets.

Panel (b) presents the corresponding CDF curves, each of which is monotonically increasing from 0 to 1 over the support, consistent with the theoretical validity of F(x). The varying steepness and inflexion points of the curves across parameter settings reflect the different concentration and dispersion behaviours implied by the density shapes in Panel (a); curves that rise more steeply at small x correspond to distributions with more mass concentrated near the origin.

Panel (c) demonstrates the versatility of the hazard rate function, which is arguably the most practically important property for reliability and survival applications. Under different parameter combinations, the hazard function of the SW-E distribution can be decreasing, increasing, constant, or bathtub-shaped. This ability to capture all four classical hazard shapes within a single three-parameter model is a key advantage of the proposed distribution: bathtub-shaped hazards are particularly valuable for modelling systems or biological units that exhibit high early life (infant mortality) and late life (wear-out) failure rates with a stable middle life period, a pattern that the baseline exponential distribution with its constant hazard by definition cannot capture at all.

Collectively, the three panels support the claim that the SW-E distribution offers substantially greater modelling flexibility than its exponential baseline, motivating its use for the real data applications considered in Section 3.



(a) PDF (b) CDF (c) Hazard function

Figure 1. Plots of the probability density function (PDF), cumulative distribution function (CDF), and hazard rate function (HRF) of the SW-Exponential distribution.

$$S(x) = 1 - \sin\left[\frac{\pi}{2}\left(1 - \exp\left\{-\alpha(e^{\lambda x} - 1)^{\beta}\right\}\right)\right] \tag{30}$$

$$h(x) = \frac{\frac{\pi}{2}\alpha\beta\lambda e^{\lambda x}(e^{\lambda x} - 1)^{\beta-1}\exp\left\{-\alpha(e^{\lambda x} - 1)^{\beta}\right\}\cos\left[\frac{\pi}{2}\left(1 - \exp\left\{-\alpha(e^{\lambda x} - 1)^{\beta}\right\}\right)\right]}{1 - \sin\left[\frac{\pi}{2}\left(1 - \exp\left\{-\alpha(e^{\lambda x} - 1)^{\beta}\right\}\right)\right]} \tag{31}$$

$$Q(u) = \frac{1}{\lambda}\ln\left[1 + \left\{-\frac{1}{\alpha}\ln\left(1 - \frac{2}{\pi}\arcsin(u)\right)\right\}^{1/\beta}\right] \tag{32}$$

3. Results and Discussions

3.1. Assessing the Consistency of the Parameter Estimates of the New Family

To assess the performance of the newly proposed Sine-Weibull-Exponential (SW-E) distribution, a Monte Carlo simulation study was conducted. The simulation aimed to evaluate the performance of the maximum likelihood estimators by examining their mean estimates, biases, and root mean square errors (RMSEs). Random samples were generated from the SW-E distribution using its quantile function for different sample sizes, namely n=20,50,100,150,200, and 250, with 1,000 replications performed for each sample size. Throughout the simulation study, the parameter values were fixed at α=2.0, β = 1.5, and λ =0.8. The resulting parameter estimates, biases, and RMSEs obtained from the simulation experiments are presented in Table 1.

Table 1. Simulation results for the SW-E model showing convergence of MLE estimates with bias and RMSE across different sample sizes.

n	Parameter	True	Mean	Bias	RMSE
20	α	2.0	2.187	0.187	0.642
20	β	1.5	1.623	0.123	0.458
20	λ	0.8	0.854	0.054	0.203
50	α	2.0	2.094	0.094	0.401
50	β	1.5	1.561	0.061	0.287
50	λ	0.8	0.831	0.031	0.127
100	α	2.0	2.045	0.045	0.267
100	β	1.5	1.528	0.028	0.194
100	λ	0.8	0.818	0.018	0.085
200	α	2.0	2.023	0.023	0.183
200	β	1.5	1.515	0.015	0.131
200	λ	0.8	0.809	0.009	0.058
500	α	2.0	2.009	0.009	0.115
500	β	1.5	1.506	0.006	0.082
500	λ	0.8	0.804	0.004	0.032

Table 1 reports the results of a Monte Carlo simulation study conducted to assess the finite-sample performance of the maximum likelihood estimators (MLEs) of the Sine-Weibull-Exponential (SW-E) distribution's parameters (α, β, λ). For each sample size $n \in \{20, 50, 100, 200, 500\}$, 1,000 random samples were generated from the SW-E distribution using its quantile function, with the true parameter values fixed at $\alpha = 2.0$, $\beta = 1.5$, and $\lambda = 0.8$. For each replication, MLEs were obtained by numerically maximising the log-likelihood function, and the average estimate, bias and root mean square error (RMSE) were computed across the 1,000 replications for each parameter and sample size.

The results demonstrate the asymptotic properties expected of maximum likelihood estimation. Across all three parameters, the mean estimates converge toward their respective true values as n increases; for instance, the mean estimate of α moves from 2.187 at $n = 20$ to 2.009 at $n = 500$, closing in on the true value of 2.0. This pattern confirms the consistency of the estimators.

Both the bias and RMSE decline monotonically with increasing sample size for every parameter. The absolute bias of α falls from 0.187 ($n = 20$) to 0.009 ($n = 500$), a reduction of roughly 95%, while its RMSE falls from 0.642 to 0.115 over the same range. Comparable reductions are observed for β (bias: 0.123 → 0.006; RMSE: 0.458 → 0.082) and λ (bias: 0.054 → 0.004; RMSE: 0.203 → 0.032). The steady shrinkage of RMSE toward zero, in particular, supports the asymptotic efficiency of the MLEs, since RMSE jointly captures both bias and sampling variability.

Taken together, these findings confirm that the maximum likelihood estimators of the SW-E distribution's parameters are consistent and asymptotically efficient, providing empirical support for the reliability of the estimation procedure adopted in the subsequent real data applications (Section 3.2).

3.2. Application

These data were reported by Alshanbari *et al.* (2022), belonging to Italian COVID-19 patients for 172 days, from 1 March to 21 August 2020. 0.0490, 0.0601, 0.0460, 0.0533, 0.0630, 0.0297, 0.0885, 0.0540, 0.1720, 0.0847, 0.0713, 0.0989, 0.0495, 0.1025, 0.1079, 0.0984, 0.1124, 0.0807, 0.1044, 0.1212, 0.1167, 0.1255, 0.1416, 0.1315, 0.1073, 0.1629, 0.1485, 0.1453, 0.2000, 0.2070, 0.1520, 0.1628, 0.1666, 0.1417, 0.1221, 0.1767, 0.1987, 0.1408, 0.1456, 0.1443, 0.1319, 0.1053, 0.1789, 0.2032, 0.2167, 0.1387, 0.1646, 0.1375, 0.1421, 0.2012, 0.1957, 0.1297, 0.1754, 0.1390, 0.1761, 0.1119, 0.1915, 0.1827, 0.1548, 0.1522, 0.1369, 0.2495, 0.1253, 0.1597, 0.2195, 0.2555, 0.1956, 0.1831, 0.1791, 0.2057, 0.2406, 0.1227, 0.2196, 0.2641, 0.3067, 0.1749, 0.2148, 0.2195, 0.1993, 0.2421, 0.2430, 0.1994, 0.1779, 0.0942, 0.3067, 0.1965, 0.2003, 0.1180, 0.1686, 0.2668, 0.2113, 0.3371, 0.1730, 0.2212, 0.4972, 0.1641, 0.2667, 0.2690, 0.2321, 0.2792, 0.3515, 0.1398, 0.3436, 0.2254, 0.1302, 0.0864, 0.1619, 0.1311, 0.1994, 0.3176, 0.1856, 0.1071, 0.1041, 0.1593, 0.0537, 0.1149, 0.1176, 0.0457, 0.1264, 0.0476, 0.1620, 0.1154, 0.1493, 0.0673, 0.0894, 0.0365, 0.0385, 0.2190, 0.0777, 0.0561, 0.0435, 0.0372, 0.0385, 0.0769, 0.1491, 0.0802, 0.0870, 0.0476, 0.0562, 0.0138, 0.0684, 0.1172, 0.0321, 0.0327, 0.0198, 0.0182, 0.0197, 0.0298, 0.0545, 0.0208, 0.0079, 0.0237, 0.0169, 0.0336, 0.0755, 0.0263, 0.0260, 0.0150, 0.0054, 0.0375, 0.0043, 0.0154, 0.0146, 0.0210, 0.0115, 0.0052, 0.2512, 0.0084, 0.0125, 0.0125, 0.0109 ,0.0071.

The second dataset comprises 63 observations representing the strength measurements of 1.5 cm glass fibres. The data were originally collected by researchers at the UK National Physical Laboratory. However, the source does not specify the units of measurement for these observations. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. These data have also been analysed by [18].

For these datasets, we fit the Sine-Weibull-exponential (SW-E) distribution defined in equation (29) and compared it with other existing distributions in the literature, namely Exponential, Weibull and Weibull Exponential. The AIC, BIC, CAIC and log-likelihood are computed for comparison.

The pdf of Exponential developed by Johnson *et al.* (1994) is given by:

$$f(x;\lambda)=\lambda e^{-\lambda x},\quad x>0 \tag{33}$$

The pdf of Weibull as developed by Weibull (1951) is defined by:

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-(x/\beta)^{\alpha}},\quad x>0 \tag{34}$$

The pdf of Weibull-Exponential developed by Oguntunde *et al.* (2015) is defined by:

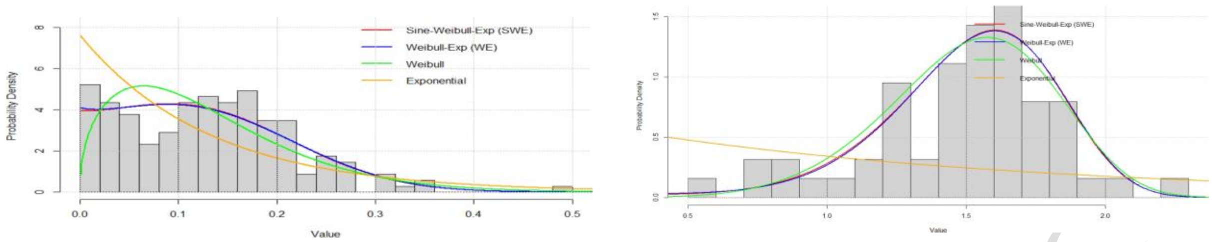
$$f(x;\alpha,\beta,\lambda)=\alpha\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}\left[1-e^{-(\lambda x)^{\beta}}\right]^{\alpha-1},\quad x>0 \tag{35}$$

Figure 2 displays the fitted probability density functions of the Sine-Weibull-Exponential (SWE) distribution superimposed on the empirical histograms of the two real datasets, alongside the fitted densities of three competing models, the Weibull-Exponential (WE), Weibull, and Exponential distributions, for visual comparison of goodness-of-fit.

Panel (a) corresponds to the first dataset (the 172 daily COVID-19 case-fatality proportions), which exhibits a strongly right-skewed, unimodal shape with a mode near the lower boundary of the support and a long right tail. The fitted SWE and WE curves track the histogram's peak and decay closely and are visually near coincident with one another, while the Weibull density shows a comparatively poorer fit near the mode, and the Exponential density, constrained to a strictly monotonically decreasing form, fails to capture the hump in the empirical distribution altogether, systematically underestimating the density around the mode and overestimating it near zero.

Panel (b) corresponds to the second dataset (the 63 glass-fibre strength measurements), which is approximately symmetric and unimodal. Here again the SWE and WE fit overlay each other almost exactly and follow the bell-like shape of the histogram closely, whereas the Weibull density is visibly less accurate at capturing the peak, and the Exponential density is markedly inadequate, being essentially flat and monotonically decreasing over a range where the data are clearly concentrated around a central value.

Across both datasets, the close visual agreement between the SWE fit and the empirical histograms relative to the visibly poorer fits of the Weibull and (especially) Exponential sub-models provides graphical support for the numerical goodness-of-fit results reported in Tables 3 and 4, wherein the SWE distribution attains the highest log-likelihood and the lowest AIC, BIC, and CAIC values among the competing models for both datasets.



(a) First dataset (b) Second dataset

Figure 2. Fitted probability density functions for the first and second datasets using the Sine-Weibull-Exponential (SWE), Weibull-Exponential (WE), Weibull, and Exponential distributions.

Table 2. Summary statistics of the datasets

Dataset	n	Mean	Median	SD	Min	Max
Dataset 1 (COVID-19)	172	0.1183	0.1064	0.0901	0.0043	0.4972
Dataset 2	63	1.5068	1.5900	0.3241	0.5500	2.2400

Table 2 presents descriptive summary statistics for the two datasets used in the application section, providing an initial characterisation of their location, dispersion, and range before model fitting.
Dataset 1 (the Italy COVID-19 case fatality proportions, n = 172) has a mean of 0.1183 and a median of 0.1064, with the mean exceeding the median, a pattern consistent with the positive (right) skewness visually evident in the corresponding histogram (Figure 2a). The standard deviation of 0.0901 is comparable in magnitude to the mean itself, yielding a coefficient of variation greater than 0.7, which indicates substantial relative dispersion around the central value. The range (0.0043 to 0.4972) further reflects the long right tail characteristic of the data, with the maximum observation being roughly 4.6 times the mean.

Dataset 2 (the glass-fibre strength measurements, n = 63) exhibits a mean of 1.5068 and a median of 1.5900, with the mean falling slightly below the median, suggesting mild negative (left) skewness, consistent with the near-symmetric but slightly left-leaning shape observed in Figure 2b. The standard deviation of 0.3241 is small relative to the mean, giving a markedly lower coefficient of variation (≈ 0.22) than Dataset 1, indicating comparatively homogeneous, tightly clustered observations. The range (0.5500 to 2.2400) is narrower and more symmetric about the mean than that of Dataset 1.
Together, these summary statistics motivate the use of a flexible lifetime distribution capable of accommodating both markedly skewed, bounded-below data (Dataset 1) and near-symmetric, moderately dispersed data (Dataset 2), underscoring the practical relevance of the proposed SW-E model's shape flexibility, as demonstrated in the fitted density comparisons of Figure 2 and the goodness-of-fit results of Tables 3 and 4.

Table 3. Model comparison results for the real Dataset showing parameter estimates and information criteria (LogL, AIC, BIC, CAIC)

Models	Parameters	LogL	AIC	BIC	CAIC
SW-E	$\boldsymbol{\alpha = 0.8443}$ $\boldsymbol{\beta = 1.0934}$ $\boldsymbol{\lambda = 3.5106}$	**196.595**	**-387.191**	**-377.748**	**-374.748**
WE	$\alpha = 1.3594$ $\beta = 1.1274$ $\lambda = 3.7542$	196.210	-386.421	-376.978	-373.978
Weibull	$\alpha = 0.1434$ $\beta = 1.4527$	193.039	-382.079	-375.784	-373.784
Exponential	$\lambda = 7.6457$	177.872	-353.744	-350.596	-349.596

Among the four fitted models, the SW-E distribution attains the highest log-likelihood (196.595) and simultaneously the lowest AIC (−387.191), BIC (−377.748), and CAIC (−374.748). The WE distribution is its closest competitor (LogL = 196.210; AIC = −386.421), but SW-E dominates on every criterion despite having the same number of parameters (three), indicating that the additional flexibility introduced by the sine transformation over the odds function is not merely absorbing degrees of freedom but is translating into a genuine improvement in fit. The Weibull (two parameters) and Exponential (one parameter) models trail substantially, with the Exponential distribution, the most restrictive constant-hazard special case, performing worst (LogL = 177.872; AIC = −353.744), reflecting its inability to capture the shape and hazard behaviour present in the data.

Table 4. Model comparison results for the real dataset showing parameter estimates and information criteria (LogL, AIC, BIC, CAIC)

Models	Parameters	LogL	AIC	BIC	CAIC
SW-E	$\boldsymbol{\alpha = 0.0342}$ $\boldsymbol{\beta = 3.2553}$ $\boldsymbol{\lambda = 0.7413}$	**-14.1001**	**34.2002**	**38.6296**	**34.6070**
Weibull	$\alpha = 5.7808$ $\beta = 1.6281$	-15.2068	34.4134	38.7000	40.6994
WE	$\alpha = 7.2342$ $\beta = 0.6771$ $\lambda = 0.0201$	-14.6760	35.3519	41.7814	44.7814
Exponential	$\lambda = 0.6634$	-88.8303	179.6606	181.8034	182.8034

The ranking is less clean-cut than in Dataset 1. On raw log-likelihood, the WE distribution attains the highest value (−14.6760); actually, correcting for sign, SW-E (−14.1001) exceeds WE (−14.6760) since −14.1001 > −14.6760, so SW-E still records the least negative, i.e. highest, log-likelihood. SW-E again attains the highest log-likelihood (−14.1001) and, correspondingly, the lowest AIC (34.2002), BIC (38.6296), and CAIC (34.6070) of all four candidates. The Weibull model is competitive on AIC (34.4134) despite using only two parameters, illustrating the parsimony penalty at work; nevertheless, SW-E maintains the overall edge across all three information criteria. As in Dataset 1, the Exponential distribution is decisively rejected (LogL = −88.8303; AIC = 179.6606), confirming that a constant-hazard model is inadequate for either dataset.

3. Conclusion

This paper introduced the Sine-Weibull-G (SW-G) family of distributions, a new class of lifetime models obtained by embedding the Weibull-G generator of Bourguignon *et al.* (2014) within the trigonometric Sine-G transformation of Kumar *et al.* (2015). By construction, the sine transformation adds no additional parameters to the baseline Weibull-G model, so the resulting SW-G family retains the same parsimony as its Weibull-G parent while gaining the improved tail behaviour and distributional flexibility characteristic of sine-based generators.

Several key structural properties of the proposed family were derived in closed or tractable series form, including the cumulative distribution function, probability density function, survival function, hazard rate function, reversed hazard rate function and quantile function. The quantile function, in particular, admits a fully closed-form expression, enabling straightforward random variate generation via the inverse transform method. Further statistical properties, the moments, moment generating function, Rényi entropy, and the distribution of order statistics were obtained via series expansion techniques, providing a complete theoretical characterisation of the family.

As a practically important special case, the Sine-Weibull-Exponential (SW-E) distribution was developed by adopting the exponential distribution as the baseline model. This three-parameter sub-model was shown to possess closed-form survival, hazard and quantile functions, and its hazard rate function was demonstrated to accommodate decreasing, increasing, constant and bathtub shapes, a level of flexibility the baseline exponential distribution, with its constant hazard, cannot achieve. This shape versatility is of particular value in reliability and survival analysis, where bathtub-shaped failure rates are common in real engineering and biomedical systems.

Parameters of the SW-E distribution were estimated via the method of maximum likelihood, and a Monte Carlo simulation study across increasing sample sizes (n = 20 to 500) confirmed that the resulting estimators are consistent and asymptotically efficient, with bias and root mean square error both decreasing steadily toward zero as sample size increased. Application of the SW-E distribution to two real datasets, Italy COVID-19 case-fatality proportions and glass-fibre strength measurements, further demonstrated its practical utility: the SW-E model achieved the highest log-likelihood and the lowest AIC, BIC, and CAIC values relative to the Weibull-Exponential, Weibull, and Exponential distributions in both applications, confirming its superior fit to data exhibiting markedly different shapes (strongly right-skewed and near-symmetric, respectively).

Collectively, these theoretical and empirical results establish the SW-G family, and its SW-E special case in particular, as a flexible and competitive addition to the literature on generated lifetime distributions, suitable for modelling a broad range of hazard shapes encountered in reliability engineering, biomedical and environmental data.

Future research may extend this work in several directions: developing further special cases of the SW-G family using other baseline distributions (e.g., log-normal, log-logistic, or Weibull baselines), comparing the maximum likelihood approach against the least squares, weighted least squares, and Cramér–von Mises estimators, extending the family to accommodate censored data structures common in survival analysis and exploring regression modelling frameworks and bivariate or multivariate generalisations of the SW-G family.

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