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ON THE FLEXIBILITY OF TYPE I HALF LOGISTIC EXPONENTIATED - INVERSE WEIBULL WITH APPLICATION TO SURVIVAL ANALYSIS

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ABSTRACT

In the field of distribution theory, statisticians have introduced new models to generalize existing ones, enhancing their flexibility and applicability across various domains. This article introduces a novel distribution called the Type I Half-Logistic Exponentiated Inverse Weibull (TIHLEtIW) Distribution, characterized by four positive parameters, extending the Inverse Weibull distribution by incorporating two additional parameters. The paper discusses several statistical properties of the TIHLEtIW distribution, including explicit expressions for the quantile function, probability-weighted moments, moments generating function, reliability function, hazard function, and order statistics. Maximum likelihood estimation is used to determine the model parameters, supported by a simulation study. The new distribution's effectiveness is demonstrated through its application to two real data sets, showing a better fit compared to other considered distributions.

Keywords: Type I Half-Logistic Exponentiated-G, Inverse Weibull distribution, Quantile function, Reliability function, Maximum likelihood, Order Statistics.

1. INTRODUCTION

All parametric statistical techniques, including inference, modeling, survival analysis, and reliability, are grounded in statistical distributions. When analyzing lifetime data, fitting it to an appropriate statistical model is a crucial step. Consequently, numerous lifespan distributions have been developed in the literature. However, most lifespan models exhibit a limited range of behaviors and often fail to provide an optimal fit for all real-world scenarios. To address this, various distribution classes have been proposed by extending traditional continuous distributions such as Exponentiated Type II Generalized

Topp-Leone-G Family of Distribution by Kolawole *et al*. (2024a), Kumaraswamy Type II Generalized Topp-Leone-G Family of Distributions with Applications by Kolawole *et al*. (2004b) and so on. This generated family of continuous distributions represents advancement in developing and enhancing classic distributions. These newly generated distributions have been extensively studied across multiple fields, offering increased flexibility for application.

The Inverse Weibull (IW) distribution is a modification of the Weibull distribution with transformation variables. Its interest is due to its flexibility and simplicity, and it can also be used to model a variety of failure characteristics. The Inverse Weibull distribution, pioneered by Keller and Kanath (1982), was designed to analyze the decay of mechanical components in survival and reliability studies. In recent years, the Inverse Weibull distribution has garnered significant attention in the literature, leading to the development of various extensions such as the Beta Inverse Weibull by Khan (2010), the Kumaraswamy-Inverse Weibull by Shahbaz *et al.,* (2012), the Reflected Generalized Beta Inverse Weibull by Elbatal *et al.*, (2016), the Topp-Leone Inverse Weibull by Abbas *et al.* (2017), the Marshall-Olkin Extended Inverse Weibull by Pakungwati *et al.,* (2018), the Odd Frechet Inverse Weibull by Fayomi (2019), the Gamma-Inverse Weibull by Abbas *et al.*, (2020) , the Extended Inverse Weibull by Alkarni *et al.*, (2020) and the modified Burr XII Inverse Weibull by Bhatti *et al.*, (2020). Bello *et al.*, (2021) introduced a new distribution family known as the Type I Half-Logistic Exponentiated-G (TIHLEt-G), which includes two additional shape parameters. For any given baseline cumulative distribution function (cdf) $H\left(x,\zeta\right)$, the TIHLEt-G family, with two positive shape parameters α and λ, has its cumulative distribution function (cdf) and probability density function (pdf) defined as follows:

$$F_{TIHLEt-G}(x;\lambda,\alpha,\zeta)=\frac{1-[1-H^{\alpha}(x;\zeta)]^{\lambda}}{1+[1-H^{\alpha}(x;\zeta)]^{\lambda}},\ x>0,\lambda,\alpha>0 \text{ and } \zeta \text{ is parameter vector} \quad (1)$$

and

$$f_{TIHLEt-G}(x;\lambda,\alpha,\zeta)=\frac{2\lambda\alpha h(x;\zeta)H^{\alpha-1}(x;\zeta)[1-H^{\alpha}(x;\zeta)]^{\lambda-1}}{\left[1+[1-H^{\alpha}(x;\zeta)]^{\lambda}\right]^{2}},\ x>0,\lambda,\alpha>0 \quad (2)$$

The cdf and pdf of the Inverse Weibull distribution are given as follows:

$$H(x;\theta,\beta)=e^{-\theta x^{-\beta}},\ x>0,\theta,\beta>0 \quad (3)$$

$$h(x;\theta,\beta)=\theta\beta x^{-\beta-1}e^{-\theta x^{-\beta}},\ x>0,\theta,\beta>0 \quad (4)$$

The aim of this paper is to develop a more flexible model by extending the two-parameter Inverse Weibull distribution. The new model is called the Type I Half Logistic Exponentiated Inverse Weibull (TIHLEtIW) distribution. We build the TIHLEtIW distribution based on the work of Bello *et al.,* (2021) and outline some key statistical properties. The paper is structured as follows: Section 2 defines the TIHLEtIW distribution. In Section 3, we present useful representations of the TIHLEtIW distribution. Section 4 derives several statistical properties, including probability-weighted moments, moments, moment generating function, quantile function, reliability function, hazard function, and order statistics. Section 5 discusses the estimation of the new model's parameters using the maximum likelihood estimation (MLE) method. In Section 6, a simulation study demonstrates the efficiency and consistency of the MLE estimates. Section 7 applies the new model to two real data sets to illustrate its practical use. Finally, Section 8 concludes the paper.

2. METHODOLOGY

Type I Half-Logistic Exponentiated Inverse Weibull (TIHLEtIW) Distribution.

In this section, we introduce a new model called the TIHLEtIW model. A random variable X is said to follow the TIHLEtIW model if its cumulative distribution function (cdf) is derived by substituting equation (3) into equation (1) as follows:

$$F_{TIHLEtIW}(x;\lambda,\alpha,\theta,\beta)=\frac{1-\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda}},\ x>0,\lambda,\alpha,\theta,\beta>0 \tag{5}$$

and its corresponding pdf is

$$f_{TIHLEtIW}(x;\lambda,\alpha,\theta,\beta)=\frac{2\lambda\alpha\theta\beta x^{-\beta-1}e^{-\theta x^{-\beta}}\left[e^{-\theta x^{-\beta}}\right]^{\alpha-1}\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right]^{2}} \tag{6}$$

where β is a scale parameter and λ,α,θ are shape parameters.

Useful Representation

In this section, we derive a useful representation for the TIHLEtIW probability density function (pdf) and cumulative distribution function (cdf). This is facilitated by the generalized binomial series, which is given by

$$\left(1+Z\right)^{-\omega}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\varpi+i-1}{i}z^{i} \qquad (7)$$

For $|z|<1$ and ω is a positive real non integer. The density function of the TIHLEtIW distribution is then obtained by using the binomial theorem (7) to (6).

$$f_{TIHLEtIW}(x;\lambda,\alpha,\theta,\beta)=2\lambda\alpha\theta\beta x^{-\beta-1}\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\sum_{i=0}^{\infty}(-1)^{i}\binom{1+i}{i}\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda(i+1)-1}$$

Now, by applying the generalized binomial theorem, we can express:

$$\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda\ i+1\ -1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\lambda(i+1)-1}{j}\left[e^{-\theta x^{-\beta}}\right]^{\alpha j}$$

Then, the pdf can be expressed as:

$$f_{TIHLEtIW}(x;\lambda,\alpha,\theta,\beta)=2\lambda\alpha\theta\beta x^{-\beta-1}\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\left[e^{-\theta x^{-\beta}}\right]^{\alpha(j+1)} \qquad (8)$$

Additionally, an expansion for $\left[F(x,\lambda,\alpha,\theta,\beta)\right]^{h}$ is provided, where h is an integer, and the binomial expansion is applied again.

$$\left[F_{TIHLEtIW}(x;\lambda,\alpha,\theta,\beta)\right]^{h}=\underbrace{\left[1-\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right]^{h}}_{A}\underbrace{\left[1+\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right]^{-h}}_{B}$$

$$A=\left[1-\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right]^{h}=\sum_{n=0}^{h}(-1)^{n}\binom{h}{n}\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda n}$$

$$B=\left[1+\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right]^{-h}=\sum_{d=0}^{h}(-1)^{d}\binom{h+d-1}{d}\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda d}$$

By combining A and B, we derive:

$$\left[F_{TIHLEtW}(x;\lambda,\alpha,\theta,\beta)\right]^{h}=\sum_{d,n=0}^{h}(-1)^{d+n}\binom{h}{n}\binom{h+d-1}{d}\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda(n+d)}$$

$$\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda(n+d)}=\sum_{q=0}^{\infty}(-1)^{q}\binom{\lambda\left(n+d\right)}{q}\left[e^{-\theta x^{-\beta}}\right]^{\alpha q}$$

The cdf can be expressed as:

$$\left[F_{TIHLEtW}(x;\lambda,\alpha,\theta,\beta)\right]^{h}=\sum_{d,n=0}^{h}\sum_{q=0}^{\infty}\left(-1\right)^{d+n+q}\binom{h}{n}\binom{h+d-1}{d}\binom{\lambda\left(n+d\right)}{q}\left[e^{-\theta x^{-\beta}}\right]^{\alpha q} \qquad (9)$$

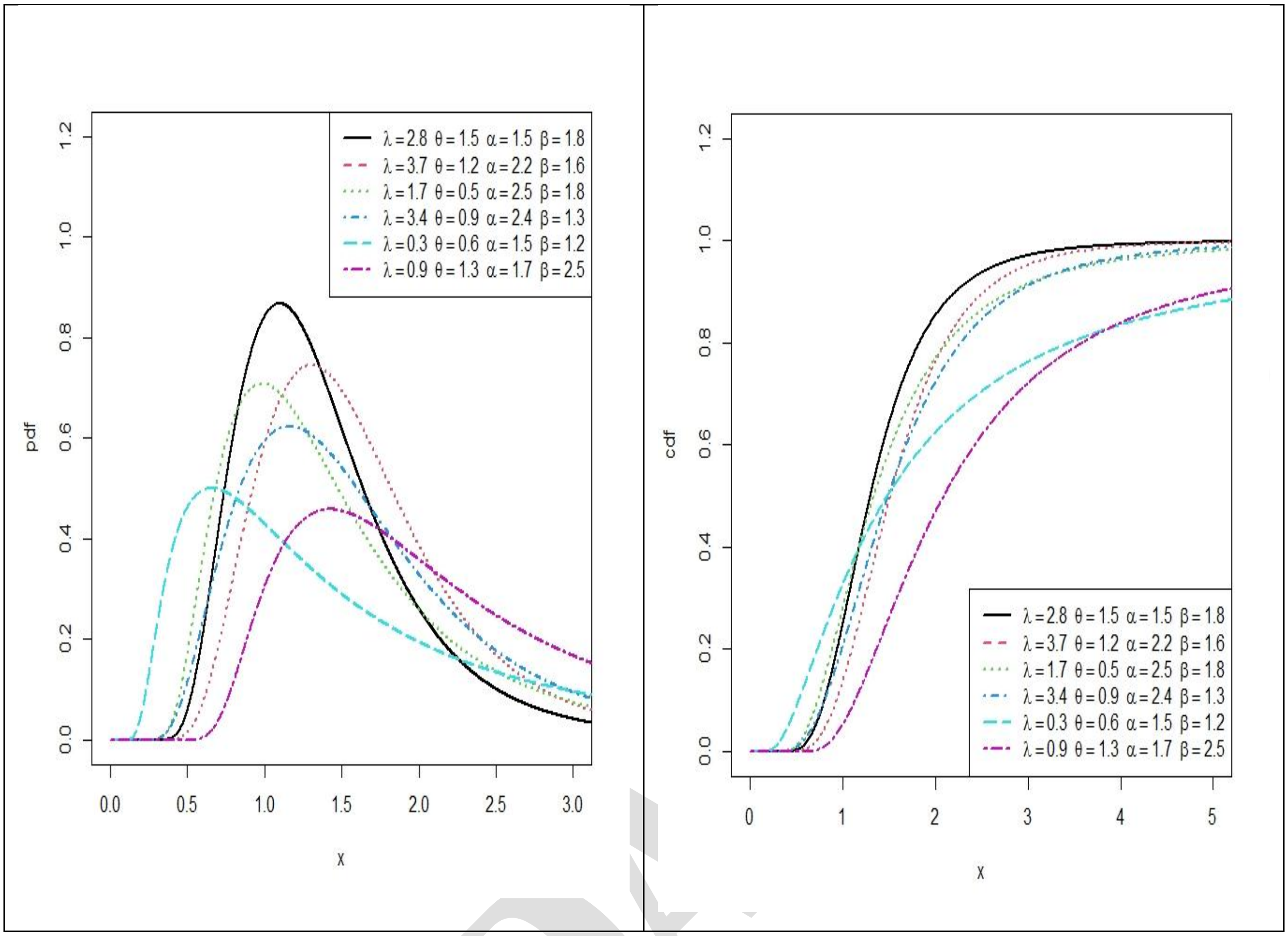

Figure 1: Plots of Pdf and Cdf of TIHLEtIW distribution for different values of parameters.

Statistical Properties

In this section, we derive several statistical properties of the new distribution.

Probability Weighted Moments

Greenwood *et al.*, (1979) introduced a class of moments called probability weighted moments (PWMs). These moments are utilized to derive inverse form estimators for the parameters and quantiles of a distribution. The PWMs, denoted by $\tau_{r,s}$, can be calculated for a random variable X using the following relationship.

$$\tau_{r,s} = E\left[X^r F(X)^s\right] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s dx \tag{10}$$

The PWMs of TIHLEtIW distribution is derive by substituting (8) and (9) into (10), and replacing h with s, as follows

$$\tau_{r,s} = 2\lambda\alpha\theta\beta x^{-\beta-1} \sum_{i,j,g=0}^{\infty} \sum_{d,n=0}^{s} (-1)^{d+n+q+i+j} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{s}{n} \binom{s+d-1}{d} \binom{\lambda(n+d)}{q} \int_0^{\infty} x^r \left[e^{-\theta x^{-\beta}}\right]^{\alpha(j+q+1)} dx \quad (11)$$

Consider the integral

$$\int_0^{\infty} x^r \left[e^{-\theta x^{-\beta}}\right]^{\alpha(j+q+1)} dx$$

Let $y = \alpha(j+q+1)\theta x^{-\beta} \Rightarrow x = \left[\frac{y}{\alpha(j+q+1)\theta}\right]^{\frac{-1}{\beta}}; dx = \frac{dy}{\alpha(j+q+1)\theta\beta x^{-\beta-1}}$

Then

$$\int_0^{\infty} \left[\frac{y}{\alpha(j+q+1)\theta}\right]^{\frac{-r}{\beta}} e^{-y} \frac{dy}{\alpha(j+q+1)\theta\beta x^{-\beta-1}} = \frac{1}{\left[\alpha(j+q+1)\right]^{\frac{r}{\beta}+1} \theta^{\frac{r}{\beta}}} \int_0^{\infty} y^{\frac{-r}{\beta}} e^{-y} dy$$

$$\int_0^{\infty} y^{\frac{-r}{\beta}} e^{-y} dy = \Gamma\left(1-\frac{r}{\beta}\right)$$

The PWMs of TIHLEtIW can be written as follows

$$\tau_{r,s} = \frac{2\lambda \sum_{i,j,q=0}^{\infty} \sum_{d,n=0}^{s} (-1)^{d+n+q+i+j} \binom{1+i}{i} \binom{\lambda(i+1)-1}{j} \binom{s}{n} \binom{s+d-1}{d} \binom{\lambda(n+d)}{q}}{\left[(j+q+1)\right]^{\frac{r}{\beta}+1} \theta^{\frac{r}{\beta}} \alpha^{\frac{r}{\beta}}} \Gamma\left(1-\frac{r}{\beta}\right) \quad (12)$$

Moments

Since moments are crucial for any statistical analysis, particularly in practical applications, we derive the r^{th} moment for the new distribution.

$$\mu_r^{'} = E(x^r) = \int_0^{\infty} x^r f(x) dx \quad (13)$$

By using the useful representation of the pdf in equation (8), we have

$$E(X^r) = 2\lambda\alpha\theta\beta x^{-\beta-1}\sum_{i,j=0}^{\infty} (-1)^{i+j}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\int_0^{\infty} x^r\left[e^{-\theta x^{-\beta}}\right]^{\alpha(j+1)} dx \tag{14}$$

Consider the integral

$$\int_0^{\infty} x^r\left[e^{-\theta x^{-\beta}}\right]^{\alpha(j+1)} dx$$

Let $k = \alpha(j+1)\theta x^{-\beta} \Rightarrow x = \left[\frac{k}{\alpha(j+1)\theta}\right]^{\frac{-1}{\beta}}; dx = \frac{dk}{\alpha(j+1)\theta\beta x^{-\beta-1}}$

Then

$$\int_0^{\infty}\left[\frac{k}{\alpha(j+1)\theta}\right]^{\frac{-r}{\beta}} e^{-k}\frac{dk}{\alpha(j+1)\theta\beta x^{-\beta-1}} = \frac{1}{[j+1]^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}}\int_0^{\infty} k^{\frac{-r}{\beta}}e^{-k}dk$$

$$\int_0^{\infty} k^{\frac{-r}{\beta}}e^{-k}dk = \Gamma\left(1-\frac{r}{\beta}\right)$$

The r^th moment for TIHLEtIW distribution can be written as follows

$$E(X^r) = \frac{2\lambda\sum_{i,j=0}^{\infty} (-1)^{i+j}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}}{[j+1]^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}\alpha^{\frac{r}{\beta}}}\Gamma\left(1-\frac{r}{\beta}\right) \tag{15}$$

The mean and variance of TIHLEtIW distribution are as follows

$$E(X) = \frac{2\lambda\sum_{i,j=0}^{\infty} (-1)^{i+j}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}}{[j+1]^{\frac{1}{\beta}+1}\theta^{\frac{1}{\beta}}\alpha^{\frac{1}{\beta}}}\Gamma\left(\frac{\beta-1}{\beta}\right) \tag{16}$$

and

$$\operatorname{var}(X)=\frac{2\lambda\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}}{[j+1]^{\frac{2}{\beta}+1}\theta^{\frac{2}{\beta}}\alpha^{\frac{2}{\beta}}}\Gamma\left(\frac{\beta-2}{\beta}\right)-\left[\frac{2\lambda\alpha\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}}{[j+1]^{\frac{r}{\beta}+1}\theta^{\frac{r}{\beta}}\alpha^{\frac{r}{\beta}}}\Gamma\left(1-\frac{r}{\beta}\right)\right]^2 \quad (17)$$

Moment generating function (mgf)
The Moment Generating Function of x is given as:

$$M_x(t)=E\left(e^{tx}\right)=\int_0^{\infty}e^{tx}f(x)dx \quad (18)$$

where the expansion of $e^{tx}=\sum_{m=0}^{\infty}\frac{t^m x^m}{m!}$

The moment generating function of TIHLEtIW distribution is given by

$$M_x(t)=\frac{2\lambda\sum_{i,j=0}^{\infty}\sum_{m=0}^{\infty}t^m(-1)^{i+j}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}}{m![j+1]^{\frac{m}{\beta}+1}\theta^{\frac{m}{\beta}}\alpha^{\frac{m}{\beta}}}\Gamma\left(1-\frac{m}{\beta}\right) \quad (19)$$

Reliability function

The reliability function, also known as the survivor function, provides the probability that a patient will survive beyond a specified period of time. It is defined as follows:

$$R(x;\lambda,\alpha,\theta,\beta)=\frac{2\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]^{\lambda}} \quad (20)$$

Hazard function

The hazard function represents the probability of an event of interest occurring within a relatively short time interval and is defined as follows:

$$T(x;\lambda,\alpha,\theta,\beta)=\frac{\lambda\alpha\theta\beta x^{-\beta-1}e^{-\theta x^{-\beta}}\left[e^{-\theta x^{-\beta}}\right]^{\alpha-1}}{\left[1+\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]\right]\left[1-\left[e^{-\theta x^{-\beta}}\right]^{\alpha}\right]} \quad (21)$$

Quantile Function
The quantile function is an essential tool for generating random variables from any continuous probability distribution, making it highly important in probability theory. For a given x , the quantile function is F(x) = u , where u follows a uniform distribution U(0,1). The TIHLEtIW distribution can be easily simulated by inverting equation (5), resulting in the quantile function Q(u) defined as:

$$x = Q(u) = \left[\frac{-1}{\theta} \log \left[1 - \left[\frac{1-u}{u+1} \right]^{\frac{1}{\lambda}} \right]^{\frac{1}{\alpha}} \right]^{\frac{-1}{\beta}} \quad (22)$$

The first quartile, median, and third quartile of the TIHLEtIW distribution are determined by setting u = 0.25, 0.5, and 0.75, respectively, in equation (22).

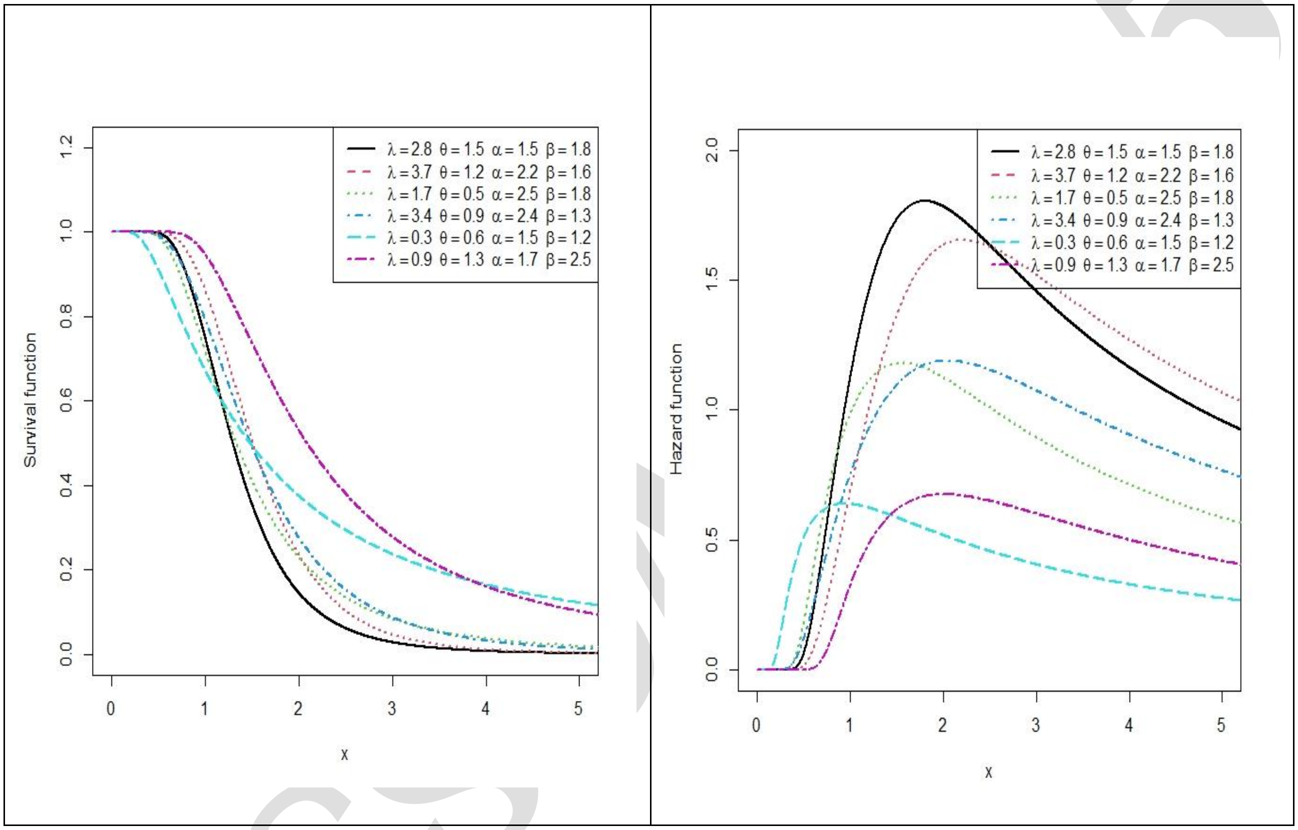

Figure 2: Plots of reliability and hazard of the TIHLEtIW distribution for different values of parameters

Order Statistics

Order statistics are extensively used in various areas of statistics, including reliability and life testing. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a continuous distribution function F(x). Suppose Let $X_1, X_2, \ldots, X_n$ are n independently distributed and continuous random variables from the TIHLEtIW distribution. Denote the cumulative distribution function (cdf) and the probability density function (pdf) of the rth order statistic $X_{\{r:n\}}$ as $F_{\{r:n\}}(x)$ and $f_{\{r:n\}}(x)$, respectively, for r = 1, 2, 3,…, n . According to David (1970), the probability density function of $X_{\{r:n\}}$ is given by:

$$f_{r:n}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \quad (23)$$

Substitute equation (8) and equation (9) into equation (23), and replace h in equation (9) with v + r-1

$$f_{r:n}(x;\lambda,\alpha,\theta,\beta)=\frac{1}{B(r,n-r+1)}2\lambda\alpha\theta\beta x^{-\beta-1}\sum_{v=0}^{n-r}\sum_{i,j,q=0}^{\infty}\sum_{d,n=0}^{v+r-1}(-1)^{i+j+q+d+n+v}\binom{n-r}{v}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{v+r-1}{n}\binom{v+r+d-2}{d}\binom{\lambda(n+d)}{q}\left[e^{-\theta x^{-\beta}}\right]^{\alpha(j+q+1)} \tag{24}$$

The equation above is called the r^{th} order statistics for the TIHLEtIW distribution.
Let r = n, in equation (24), then the probability density function of the maximum order statistics of TIHLEtIW distribution is

$$f_{n:n}(x;\lambda,\alpha,\theta,\beta)=2\lambda\alpha\theta\beta n x^{-\beta-1}\sum_{i,j,q=0}^{\infty}\sum_{d,n=0}^{v+n-1}(-1)^{i+j+q+d+n+v}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{v+n-1}{n}\binom{v+n+d-2}{d}\binom{\lambda(n+d)}{q}\left[e^{-\theta x^{-\beta}}\right]^{\alpha(j+q+1)} \tag{25}$$

Also, let r = 1 in equation (24), then the probability density function of the minimum order statistics of TIHLEtIW distribution is

$$f_{1:n}(x;\lambda,\alpha,\theta,\beta)=2\lambda\alpha\theta\beta n x^{-\beta-1}\sum_{v=0}^{n-1}\sum_{i,j,q=0}^{\infty}\sum_{d,n=0}^{v}(-1)^{i+j+q+d+n+v}\binom{n-1}{v}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{v}{n}\binom{v+d-1}{d}\binom{\lambda(n+d)}{q}\left[e^{-\theta x^{-\beta}}\right]^{\alpha(j+q+1)} \tag{26}$$

Parameter Estimation

In this section, we investigate the use of maximum likelihood estimation (MLE) to determine the unknown parameters of the TIHLEtIW distribution using complete data. MLEs are advantageous as they allow for the calculation of confidence intervals and provide straightforward approximations that perform well with finite samples. In distribution theory, the resulting MLE approximations can be handled easily, either analytically or numerically. Let $x_1, x_2, x_3, \ldots, x_n$ denote a random sample of size n from the TIHLEtIW distribution. Then, the likelihood function based on the observed sample for the vector of parameters $(\lambda, \alpha, \theta, \beta)^T$ is given by:

$$\log(\phi)=n\log(2)+n\log(\lambda)+n\log(\alpha)+n\log(\theta)+n\log(\beta)-(\beta-1)\sum_{i=1}^{n}\log(x_i)-\theta\alpha\sum_{i=1}^{n}x_i^{-\beta}+(\lambda-1)\sum_{i=1}^{n}\log\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]-2\sum_{i=1}^{n}\log\left[1+\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right] \tag{27}$$

The components of score vector $\Delta L(\phi)=\left(\frac{\partial L(\phi)}{\partial\lambda},\frac{\partial L(\phi)}{\partial\alpha},\frac{\partial L(\phi)}{\partial\theta},\frac{\partial L(\phi)}{\partial\beta}\right)^T$ are given as

$$\frac{\partial L(\phi)}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]-2\sum_{i=1}^{n}\frac{\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]^{\lambda}\log\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]}{\left[1+\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right]} \tag{28}$$

$$\frac{\partial L(\phi)}{\partial \alpha}=\frac{n}{\alpha}-\theta\sum_{i=1}^{n} x_i^{-\beta}-(\lambda-1)\sum_{i=1}^{n}\frac{\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\log\left[e^{-\theta x_i^{-\beta}}\right]}{\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]}$$
$$+2\sum_{i=1}^{n}\frac{\lambda\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]^{\lambda-1}\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\log\left[e^{-\theta x_i^{-\beta}}\right]}{\left[1+\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right]} \quad (29)$$

$$\frac{\partial L(\phi)}{\partial \theta}=\frac{n}{\theta}-\alpha\sum_{i=1}^{n} x_i^{-\beta}+(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha-1}e^{-\theta x_i^{-\beta}}x_i^{-\beta}}{\,}$$
$$+2\sum_{i=1}^{n}\frac{\lambda\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]^{\lambda-1}\alpha\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha-1}e^{-\theta x_i^{-\beta}}x_i^{-\beta}}{\left[1+\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right]} \quad (30)$$

$$\frac{\partial L(\phi)}{\partial \beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\log(x_i)-\theta\alpha\sum_{i=1}^{n} x_i^{-\beta}\log(x_i)+(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha-1}e^{-\theta x_i^{-\beta}}\theta x_i^{-\beta}\log(x_i)}{\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]}$$
$$+2\sum_{i=1}^{n}\frac{\lambda\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]^{\lambda-1}\alpha\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha-1}e^{-\theta x_i^{-\beta}}\theta x_i^{-\beta}\log(x_i)}{\left[1+\left[1-\left[e^{-\theta x_i^{-\beta}}\right]^{\alpha}\right]^{\lambda}\right]} \quad (31)$$

The MLEs are derived by setting $\frac{\partial L(\phi)}{\partial \lambda},\frac{\partial L(\phi)}{\partial \alpha},\frac{\partial L(\phi)}{\partial \theta}$ and $\frac{\partial L(\phi)}{\partial \beta}$ to zero and solving the resulting equations simultaneously. Since these equations cannot be solved analytically, numerical methods must be used to find the solutions.

3. Simulation Study

In this section, a numerical analysis is conducted to evaluate the performance of MLE for TIHLEtIW Distribution.

Table 1: MLEs, biases and RMSE for some values of parameters

		(0.6,2,1,1.7)			(0.7,3,1.1,2.2)		
n	Parameters	Estimated Values	Bais	RMSE	Estimated Values	Bais	RMSE
	λ	0.6832 2.064	0.0832 0.	0.4126 0.	0.8018 3.050	0.1018 0.	0.4890 0.
20	α	8 1.3700 2.03	0648 0.3	5759 0.8	5 1.2753 2.57	0505 0.1	6789 0.7
	θ	46	700 0.33	951 0.86	06	753 0.37	768 0.98

	β		46	53		06	34
	λ	0.6317 2.045	0.0317 0.	0.2785 0.	0.7404 3.026	0.0404 0	0.3200 0.
50	α	0 1.1618 1.90	0450 0.1	4083 0.5	9 1.1415 2.42	.0269 0.0	4714 0.4
	θ	84	618 0.20	590 0.64	94	415 0.22	292 0.74
	β		84	61		94	28
100	λ	0.6021 2.021	0.0021 0.	0.1861 0.	0.7145 3.021	0.0145 0	0.2290 0.
	α	6 1.0557 1.83	0216 0.0	3191 0.3	1 1.1206 2.34	.0211 0.0	3553 0.2
	θ	61	557 0.13	109 0.44	61	206 0.14	757 0.53
	β		61	55		61	87
150	λ	0.6066 2.027	0.0066 0.	0.1610 0.	0.7101 3.019	0.0101 0	0.1900 0.
	α	2 1.0367 1.78	0272 0.0	3100 0.2	3 1.1123 2.30	.0193 0.0	3187 0.2
	θ	30	367 0.08	617 0.36	40	123 0.10	299 0.46
	β		30	35		40	86
200	λ	0.6052 2.028	0.0052 0.	0.1364 0.	0.7098 3.025	0.0098 0	0.1677 0.
	α	5 1.0183 1.75	0285 0.0	2645 0.2	8 1.1031 2.27	.0258 0.0	2901 0.2
	θ	85	183 0.05	031 0.29	61	031 0.07	236 0.39
	β		85	48		61	43
250	λ	0.6019 2.038	0.0019 0.	0.1233 0.	0.7100 3.023	0.0100 0	0.1499 0.
	α	2 1.0073 1.75	0382 0.0	2509 0.1	4 1.1010 2.25	.0234 0.0	2408 0.2
	θ	37	073 0.05	808 0.28	24	010 0.05	134 0.34
	β		37	43		24	49

The table above demonstrates that as the sample size increases, both the biases and RMSEs approach zero, and the estimates converge to the true values. This indicates that the estimates are efficient and consistent.

4. Applications to Real Data

In this section, we fit the TIHLEtIW distribution to two real data sets and conduct a comparative study with several other distributions. These include the Extended Inverse Weibull (TIHLIW) Distribution by Alkarni *et al.*, (2020), the Marshall-Olkin Extended Inverse Weibull (MOIW) Distribution by Pakungwati *et al.*, (2018), the generalized Inverse Weibull (GIW) distribution by De Gusmao *et al.*, (2011), the Kumaraswamy–Inverse Weibull (KIW) Distribution by Shahbaz *et al.*, (2012), and the

Inverse Weibull (IW) distribution by Keller and Kanath (1982). This comparison serves to illustrate the performance of the TIHLEtIW distribution.

The TIHLIW distribution developed by Alkarni *et al.*, (2020) has pdf defined as:

$$f(x;\alpha,\lambda,\beta)=\frac{2\lambda\alpha\beta x^{-\beta-1}\exp\left(-\alpha x^{-\beta}\right)\left(1-\exp\left(-\alpha x^{-\beta}\right)\right)^{\lambda-1}}{\left[1+\left(1-\exp\left(-\alpha x^{-\beta}\right)\right)^{\lambda}\right]^{2}} \tag{32}$$

The MOIW distribution developed by Pakungwati *et al.*, (2018) has pdf defined as:

$$f(x;\beta,\alpha,\theta)=\frac{\alpha\beta\theta^{-\beta}x^{-\beta-1}\exp\left(-(x\theta)^{-\beta}\right)}{\left(\alpha-(\alpha-1)\exp\left(-(x\theta)^{-\beta}\right)\right)^{2}} \tag{33}$$

The GIW distribution proposed by De Gusmao *et al.*, (2011) has pdf given as:

$$f(x;\alpha,\lambda,\beta)=\lambda\beta\alpha^{\beta}x^{-(\beta+1)}\exp\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right) \tag{34}$$

The KIW Distribution proposed by Shahbaz *et al.*, (2012) has pdf given as:

$$f(x;\lambda,\alpha,\theta,\beta)=\frac{\lambda\theta\alpha\beta}{x^{\beta+1}}\exp\left(-\frac{\lambda\alpha}{x^{\beta}}\right)\left(1-\exp\left(-\frac{\lambda\alpha}{x^{\beta}}\right)\right)^{\theta-1} \tag{35}$$

The IW distribution developed by Keller and Kanath (1982) has pdf defined as:

$$f(x;\alpha,\lambda)=\alpha\lambda x^{-\lambda-1}\exp\left(-\alpha x^{-\lambda}\right) \tag{36}$$

The two datasets used in the application illustrate the flexibility, applicability, and superior fit of the newly proposed distribution in empirically modeling the data, compared to the aforementioned comparator distributions. All calculations were performed using the R programming language.

Data set 1

Data set 1 has been used by Hosseini *et al.,* (2018) and represents the sum of skin folds in 202 athletes collected at the Australian Institute of Sports as.

28.0, 98, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3, 109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8, 80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0, 65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3, 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8, 47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9, 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2,101.1, 47.5, 46.2, 38.2, 49.2, 49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6, 73.9, 43.5, 61.8, 88.9, 31.0, 37.6,52.8, 97.9, 111.1, 114.0, 62.9, 36.8, 56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67, 41.6, 34.8, 61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6, 48, 61.2, 171.1, 113.5, 148.9, 49.9, 59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6, 76.8, 99.8, 80.1, 57.9, 48.4, 41.8, 44.5, 43.8, 33.7, 30.9, 43.3, 117.8, 80.3, 156.6, 109.6, 50.0, 33.7, 54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5, 115.9, 98.5, 54.6, 50.9, 44.7, 41.8, 38.0, 43.2, 70.0, 97.2, 123.6, 181.7, 136.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6, 46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3, 54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9.

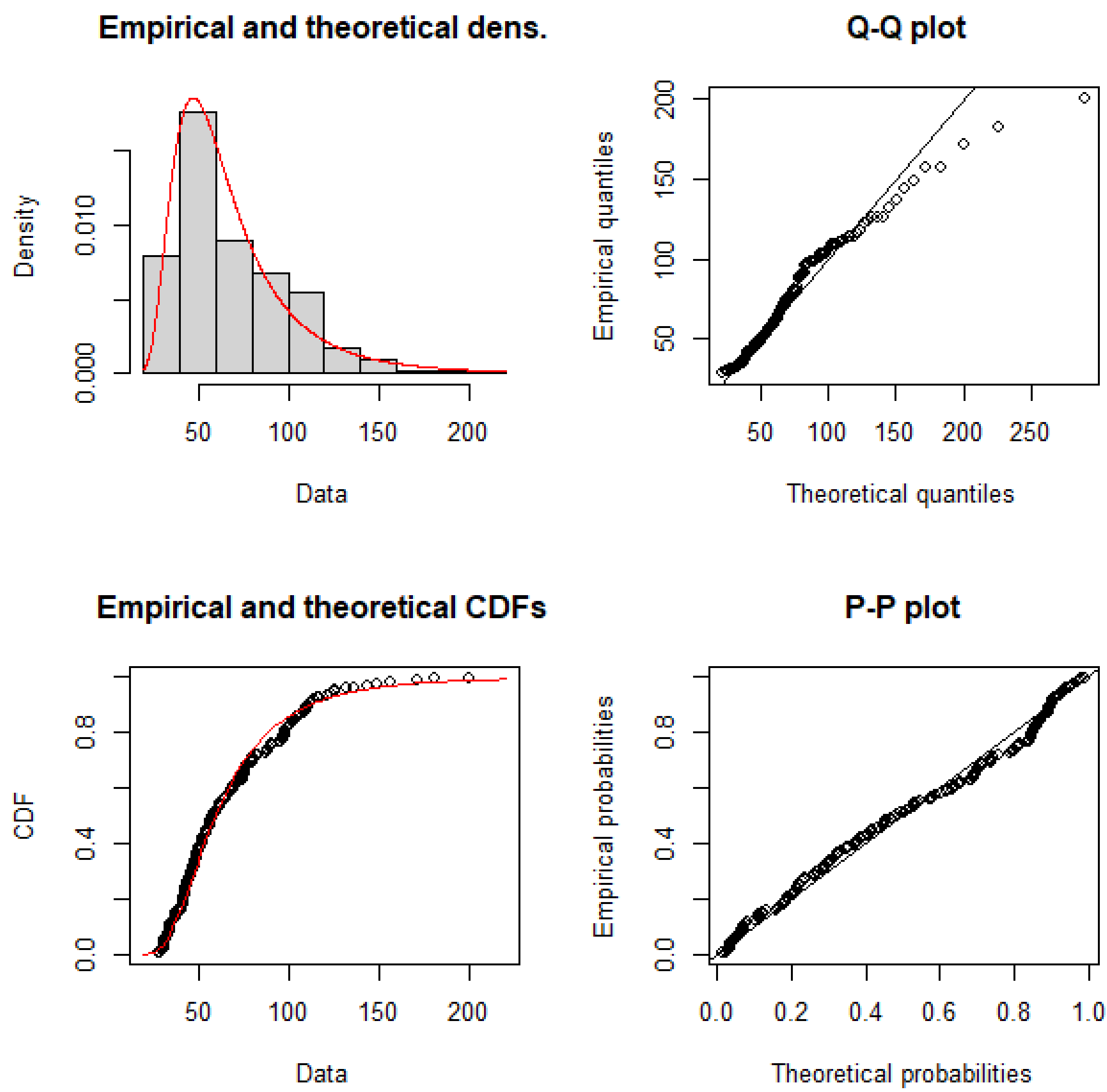

Figure 3: Empirical and theoretical pdfs and cdfs, Q-Q and P-P plots for data set 1

Table 2: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

Distributions	α	λ	θ	β	LL	AIC
TIHLEtIW	7.7960	2.1793	0.3267	1.8581	- 954.8037	1917.607
TIHLIW	4.1136	4.5666	-	0.5454	- 983.9489	1973.898
KIW	6.7487	8.1090	6.7334	0.7083	- 976.3673	1960.735

GIW	10.8781	10.6261	-	1.6681	- 991.553	1989.106
MOIW	2.9808	0.0260	3.6122	-	- 956.8616	1919.723
IW	2.1719	2.6171	-	-	-965.8263	1937.653

Table 2 presents the maximum likelihood estimation results for the parameters of the newly proposed distribution and five comparator distributions. According to the goodness-of-fit measure, the new distribution achieved the lowest AIC value, with the MOIW distribution closely following. Also, the visual inspection of the empirical and theoretical pdfs and cdfs, as well as the Q-Q and P-P plots as shown in Figure 3, further confirms the superior performance of the proposed distribution. Consequently, the new proposed distribution offers the best fit for the sum of skin folds in the athlete data set among the distributions considered.

Data set 2

Data set 2 was given by Lee (1992) and it represents the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital in a period from 1929 to 1938. The data set is as follows:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 1.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

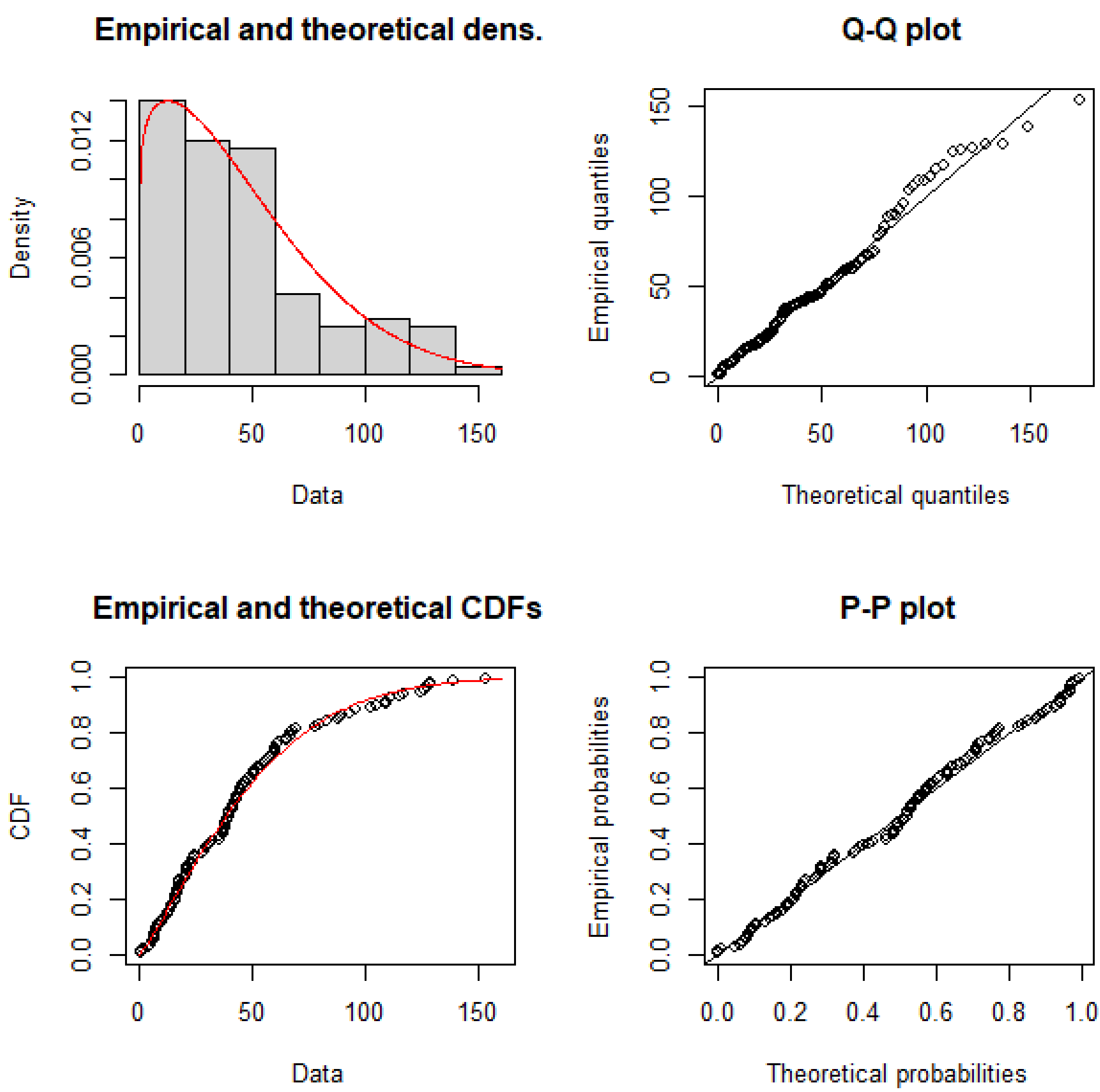

Figure 4: Empirical and theoretical pdfs and cdfs, Q-Q and P-P plots for data set 2

Table 3: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

Distributions	α	λ	θ	β	LL	AIC
TIHLEtIW	0.1139	4.7925	7.4841	0.2679	- 588.4393	1184.879
TIHLIW	8.1545	10.6049	-	0.3669	-594.8777	1195.755
KIW	4.4317	9.2653	0.2583	0.3591	-600.972	1209.944
GIW	2.2157	3.7626	-	0.6524	-636.794	1279.588
MOIW	4.2565	-	0.2232	0.8709	-627.0566	1260.113

IW	6.3232	0.6523	-	-	-636.794	1277.588

Table 3 presents the results of the maximum likelihood estimation for the parameters of the TIHLEtIW distribution and five comparator distributions. According to the goodness-of-fit statistic AIC, the new distribution reported the lowest AIC value, indicating it as the best fit for the breast cancer patient data. Also, the visual inspection of the empirical and theoretical pdfs, cdfs, and Q-Q and P-P plots, as shown in Figure 4, further reaffirms the fit and flexibility of the new distribution to the set of data considered.

5. Conclusion

In our study, we introduce and analyze a novel distribution termed the Type I Half-Logistic Exponentiated Inverse Weibull Distribution, building upon the distribution family proposed by Bello *et al.*, (2021). We thoroughly investigate several statistical properties of this new distribution, including its explicit quantile function, probability weighted moments, moments, generating function, reliability function, hazard function, and order statistics. Parameter estimation is conducted using the maximum likelihood technique, a robust method that allows us to derive accurate estimates for the distribution's parameters. We validated the performance of the proposed distribution through simulations, demonstrating the efficacy of the new model. To further underscore its utility, we apply the new distribution to analyze two real-world datasets. Our findings indicate that the Type I Half-Logistic Exponentiated Inverse Weibull Distribution outperforms existing models considered in terms of flexibility and applicability across various domains. This suggests its potential for effective data modeling in diverse practical applications.

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