



AN EMPIRICAL COMPARISON OF THE ODD RAYLEIGH LOG-LOGISTIC DISTRIBUTION WITH OTHER LIFETIME MODELS USING REAL DATASETS

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ABSTRACT

The modelling of survival and reliability data often requires flexible probability distributions capable of capturing a wide range of hazard rate behaviours. Classical models such as the log-logistic and exponential distributions may be limited in their ability to accommodate complex real-world patterns, particularly those involving heavy tails or non-monotonic hazard rates. This study evaluates the Odd Rayleigh Log-Logistic distribution (ORLLD), a novel extension designed to model survival data with enhanced flexibility in tail behaviour, hazard rates, and time-to-event outcomes. The ORLLD, which integrates the strengths of the log-logistic and Rayleigh distributions while introducing additional flexibility through the concept of odd function is characterized by a unique parameterization that allows for greater adaptability in modelling skewness and heavy tails. The key mathematical properties of the ORLLD including its probability density function (PDF), cumulative distribution function (CDF), quantile function, survival function, and hazard rate have been previously derived. The performance of the ORLLD is evaluated through Monte Carlo simulation under various sample size scenarios, employing both Maximum Likelihood Estimation (MLE) and Maximum Product of Spacings (MPS) techniques. Furthermore, the versatility of the model is demonstrated through its application to two real-life lifetime datasets. The results suggest that the ORLLD serves as a robust alternative to traditional survival models, offering improved fit and predictive accuracy in survival analysis.

Keywords: hazard rate, log-logistic, Rayleigh, maximum product of spacings, Monte Carlo simulation

1. INTRODUCTION

Probability distributions play a pivotal role in statistical modelling, providing the mathematical foundation for analyzing and interpreting data across a wide range of disciplines. Over the years, the development of new distributions has been driven by the need to model increasingly complex real-world phenomena that cannot be adequately captured by classical models. The log-logistic distribution is a popular choice for modelling survival data with non-monotonic hazard rates. Its hazard function can increase initially, reach a peak, and then decrease, making it suitable for datasets with complex hazard rate patterns (Onyeagu & Chikezie, 2021; Bennett, 1983).

The Rayleigh and log-logistic distributions are frequently employed in reliability and survival analysis, yet their rigidity poses significant challenges when modeling real-world data. The Rayleigh distribution, with its strictly increasing hazard rate and single-parameter structure, fails to capture non-monotonic failure patterns (Nadarajah & Kotz, 2006). On the other hand, the log-logistic distribution, while flexible for unimodal hazards, cannot model constant or bathtub-shaped hazards and struggles with heavy-tailed

data, which exhibit skewed and extreme values (Dey *et al.*, 2018). These limitations hinder accurate risk assessment and reliability predictions.

The Odd Rayleigh Log-Logistic Distribution (ORLLD) is critically important because it provides a highly flexible model capable of capturing complex real-world patterns such as heavy-tailed, skewed, or non-monotonic hazard behaviors that traditional survival models like the log-logistic or Weibull often fail to represent accurately. Its practical application is demonstrated through superior performance in modeling time-to-event data, offering statisticians, engineers, and medical researchers a more robust tool for enhancing predictive accuracy in fields such as reliability engineering for analyzing failure times, biomedical research for estimating patient survival rates, and any domain requiring the analysis of complex lifetime data.

This study, thus provide an empirical comparison of the Odd Rayleigh log-logistic distribution with other lifetime models using real datasets. This distribution can be used to model complex hazard rate data, censored data and heavy-tailed data.

2. LITERATURE REVIEW

Several literature have empirically explore the use of Rayleigh and log-logistic distributions in modelling different phenomena, detailing the methodology, findings, and applications across diverse disciplines. For example, Harrison, Smith & Patel (2020) used the Rayleigh distribution to model atmospheric turbulence affecting optical communication systems. By analyzing data from ground-based optical communication systems, they demonstrated that Rayleigh fading models could accurately describe the fluctuations in signal strength due to atmospheric turbulence, improving communication reliability. Onyeagu & Chikezie (2021) introduced a modified version of the log-logistic distribution for modelling financial data, called Modified log-logistic Distribution for Modeling Financial Data. The model was applied to stock market data and found it to be more flexible than the traditional log-logistic distribution. Khan, Ahmed & Zubair (2021) applied the Rayleigh distribution to model the intensity of rainfall in tropical regions. The study found that the Rayleigh distribution accurately described the intensity and duration of rainfall events, offering a statistical framework for hydrological modeling and water reeling the motion of vessels in various sea conditions, they found that Rayleigh distribution effectively represented the motion data, assisting in the design of more stable vessels under harsh environmental conditions. Li, Wang & Zhang (2022) used the Rayleigh distribution to model the failure times of electrical components, including semiconductors. Their study indicated that the Rayleigh distribution accurately captured the life cycle of components subject to temperature and voltage fluctuations, which is crucial for the design of long-lasting electronic devices. Cheng & Zhang (2022) applied the log-logistic distribution to analyze the failure times of industrial machinery. Their study focused on the failure data of machines used in manufacturing plants, where failure times often exhibit non-monotonic hazard rates. The log-logistic distribution provided a better fit for the data compared to the Weibull distribution, which is commonly used in reliability analysis. Their research emphasized the importance of using the log-logistic model for machinery that experiences both early-life failures and long-tail life cycles, making it a crucial tool for preventative maintenance planning in industrial settings. Abdullahi *et al.* (2023) presented a novel class (family) of statistical distributions termed Odd Transmuted Rayleigh-X (OTRX) that was created through a transform-transformer (T-X) approach. The study showed that the proposed distribution outperformed its competitors when using two different real-world datasets. Furthermore, they recommended that the proposed distribution can be applied to any skewed dataset. Smith & Johnson (2023) conducted a comprehensive analysis of lifetime data modelling, focusing on electronic components used in industrial applications. They compared several

statistical distributions, including the exponential, Weibull, and log-logistic distributions. Their empirical study showed that the log-logistic distribution provided the best fit for the data in terms of model selection criteria like Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The study highlighted the log-logistic distribution's ability to better capture the non-monotonic hazard rate observed in the failure patterns of the electronic components. Panitantrak, Ishaq, Abiodun, Daud & Suleiman (2025) developed the Maxwell-Log Logistic (Max-LL) distribution. To assess the distribution's adaptability, two lifetime datasets were applied to it: global COVID-19 mortality rates (for nations with more than 100,000 cases) and Canadian COVID-19 mortality rates. The Maxwell-Log Logistic distribution outperformed other distributions on both datasets, as evidenced by several accuracy measures. They opined that the proposed distribution is the best fit for COVID-19 mortality rate data in Canada and around the world.

3. PROBABILITY DISTRIBUTIONS

In this section, the probability density function (PDF) and cumulative distribution function (CDF) of the ORLLD model is presented. Likewise, the PDFs and CDFs of competing models are also presented. A brief discussion of the data used for empirical analysis is also presented.

3.1 The Odd Rayleigh Log Logistic Distribution (ORLLD)

Garba, David, Tasi'u and Zakari (2025) defined the probability density function and cumulative distribution function of the ORLLD respectively as:

$$f(x; \theta, \alpha, \beta) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{2\beta-1}}{\theta^2} \exp\left\{-\frac{1}{2\theta^2}\left(\frac{x}{\alpha}\right)^{2\beta}\right\}; \quad x, \theta, \alpha, \beta > 0 \quad (1)$$

and

$$F(x; \theta, \alpha, \beta) = 1 - \exp\left(-\frac{1}{2\theta^2}\left(\frac{x}{\alpha}\right)^{2\beta}\right); \quad x, \theta, \alpha, \beta > 0 \quad (2)$$

Table 1 gives Monte Carlo simulation of the ORLLD through maximum likelihood estimation (MLE) and maximum product of spacings (MPS). The study was carried out across various sample sizes, specifically: $n = 25, 50, 100, 250, 500$ with 1,000 replications for each sample size. To determine the performance of the MLE and MPS, the bias and Root Mean Square Error (RMSE) have been computed for the parameter combination.

Table 1: Simulation output for ORLLD for $\alpha = 1, \beta = 0.5$ and $\theta = 0.8$

N	Parameters	MLE		MPS	
		Bias	RMSE	Bias	RMSE
25	α	-0.5658	0.3363	-0.5427	0.3101
	β	0.0533	0.0284	-0.0423	0.0221
	θ	-0.5556	0.4716	-0.587	0.4794
50	α	-0.5234	0.2783	-0.5122	0.2665
	β	0.0235	0.0106	-0.0316	0.0097
	θ	-0.495	0.354	-0.524	0.3677
100	α	-0.4975	0.2484	-0.4952	0.246
	β	0.0106	0.0049	-0.0206	0.0049
	θ	-0.5278	0.3386	-0.548	0.3538
250	α	-0.4873	0.2376	-0.4877	0.238
	β	0.0022	0.0018	-0.0125	0.0019
	θ	-0.5339	0.3107	-0.5449	0.3212
500	α	-0.4847	0.2349	-0.4848	0.2351
	β	0.0009	0.0009	-0.0072	0.001
	θ	-0.5335	0.2965	-0.5394	0.3025

3.2 Data

Data 1 represents the failure times of the air conditioning system (in hours) of an airplane. The data set was given by Linhart and Zucchini (1986), it has been used by Sule *et al.* (2020). The data set is presented as:

23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95

Data 2 is obtained from Patil and Rao (1994), and represents the distances from the transect line for the 68 stakes detected in walking (L=1000m) and searching (w=20m) on each side of the line. The measurements are:

2.0, 0.5, 10.4, 3.6, 0.9, 1.0, 3.4, 2.9, 8.2, 6.5, 5.7, 3.0, 4.0, 0.1, 11.8, 14.2, 2.4, 1.6, 13.3, 6.5, 8.3, 4.9, 1.5, 18.6, 0.4, 0.4, 0.2, 11.6, 3.2, 7.1, 10.7, 3.9, 6.1, 6.4, 3.8, 15.2, 3.5, 3.1, 7.9, 18.2, 10.1, 4.4, 1.3, 13.7, 6.3, 3.6, 9.0, 7.7, 4.9, 9.1, 3.3, 8.5, 6.1, 0.4, 9.3, 0.5, 1.2, 1.7, 4.5, 3.1, 3.1, 6.6, 4.4, 5.0, 3.2, 7.7, 18.2, 4.1

4. ANALYSIS

This section demonstrates the practical utility of the ORLLD through an application to a real dataset. We conduct a comparative analysis, fitting the ORLLD against key competitor models; log logistic distribution (LLD) Johnson *et al.* (1994), Transmuted log-logistic distribution (TLLD) by Aryal (2013) and The Kumaraswamy-log-logistic distribution (KumLLD) by De Santana *et al.* (2012) as comparator distributions for illustrative purposes. The PDFs and CDFs of the comparators are given as follows: The PDF and CDF of LLD are given in equations (3) and (4) respectively as;

$$m(x; \alpha, \beta) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1 + \left(\frac{x}{\alpha}\right)^\beta\right)^2}; \quad x, \alpha, \beta > 0 \tag{3}$$

$$M(x; \alpha, \beta) = \frac{\left(\frac{x}{\alpha}\right)^\beta}{1 + \left(\frac{x}{\alpha}\right)^\beta}; \quad x, \alpha, \beta > 0 \tag{4}$$

The PDF and CDF of TLLD are given in equations (5) and (6) respectively as;

$$f(x; \alpha, \beta, \lambda) = \frac{\beta \alpha^\beta x^{\beta-1} [(1 + \lambda)(\alpha^\beta + x^\beta) - 2\lambda x^\beta]}{(\alpha^\beta + x^\beta)^3} \quad x, \alpha, \beta, \lambda > 0 \tag{5}$$

$$F(x; \alpha, \beta, \lambda) = \frac{(1 + \lambda)\alpha^\beta x^\beta + x^{2\beta}}{(\alpha^\beta + x^\beta)^2} \quad x, \alpha, \beta, \lambda > 0 \tag{6}$$

The PDF and CDF of KumLLD are given in equations (7) and (8) respectively as;

$$f(t; a, b, \alpha, \gamma) = \frac{ab\gamma}{\alpha^{a\gamma}} t^{a\gamma-1} \left[1 + \left(\frac{t}{\alpha}\right)^\gamma\right]^{-(a+1)} \left\{1 - \left[1 - \frac{1}{1 + \left(\frac{t}{\alpha}\right)^\gamma}\right]^a\right\}^{b-1} \quad t, a, b, \alpha, \gamma > 0 \tag{7}$$

$$F(t; a, b, \alpha, \gamma) = \left\{1 - \left[1 - \frac{1}{1 + \left(\frac{t}{\alpha}\right)^\gamma}\right]^a\right\}^b \quad t, a, b, \alpha, \gamma > 0 \tag{8}$$

Table 2 gives the maximum likelihood estimates, log likelihoods and discrimination criteria for all fitted models of data on Failure time of air condition system of airplane.

Table 2: Estimates and discrimination criteria dataset 1

Distributions	α	β	θ	LL	AIC
LLD	29.30466	1.301296		-155.184	314.3687
TLLD	29.30409	1.301231	-9.141082	-152.184	310.3687
KumLLD	30.53105	1.316538	9.666532	-152.183	310.3669
ORLLD	32.80170	0.426791	2.511243	-151.937	309.8738

Based on the Akaike Information Criterion (AIC) values in Table 2, the ORLLD (AIC = 309.8738) is identified as the best-performing model for the airplane air conditioning system dataset. Its lowest AIC value confirms its greater flexibility and better fit compared to the other three competitor distributions. A visual comparison of the goodness of fit test in Figure 1 also confirmed the superiority of the proposed models over some comparators.

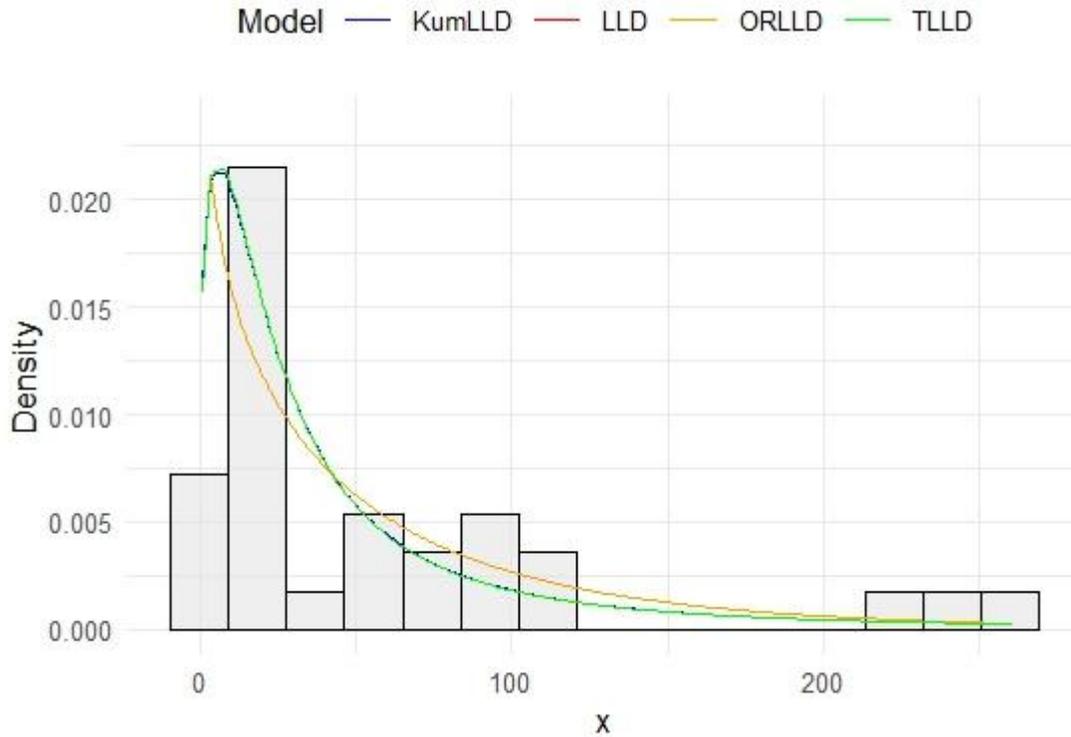


Figure 1: Fitted densities for KumLLD, LLD, ORLLD and TLLD on data set 1

Table 3 presents the parameter estimates, log-likelihood values, and goodness-of-fit statistics for the models fitted to the dataset. This dataset consists of the detection distances from the transect line for all 68 stakes found during the walking search on both sides of the line.

Table 3 Estimates and goodness of fits of dataset 2

Distributions	α	β	θ	LL	AIC
LLD	67.356012	1.6846793	-	-195.6748	395.3496
TLLD	58.045557	1.4819090	0.9996604	-189.1748	384.3497
KumLLD	60.244215	3.8521716	0.2421899	-186.3957	378.7914
ORLLD	66.370549	0.6123837	0.1661725	-186.1698	378.3396

To assess the flexibility of the models proposed, Table 3 shows that the ORLLD has the lowest AIC value of 378.3396 indicating that it fits the distance of transect line data set better than the other three comparators considered. A visual assessment of the goodness of fit test in Figure 2 also confirmed the superiority of the proposed models over some comparators.

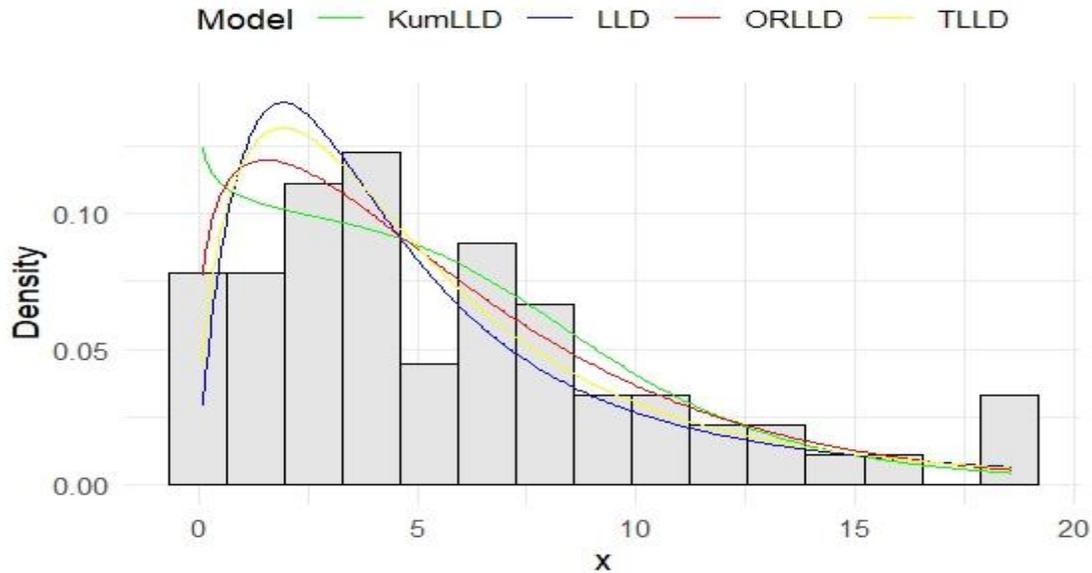


Figure 1: Fitted densities for KumLLD, LLD, ORLLD and TLLD on data set 2

5. CONCLUSION

In this study, an empirical analysis to compare the performance of the odd Rayleigh Log-logistic distribution with those of its comparators has been carried out. The comparators include; log logistic distribution (LLD), Transmuted log-logistic distribution (TLLD), and Kumaraswamy-log-logistic distribution (KumLLD). Two real-life datasets were used to assess the applicability of the compound distribution. The result indicated that ORLLD model is more adequate and best captures the real-life data sets than the competing models.

Thus, the odd Rayleigh Log-logistic distribution is a suitable model for dataset on failure time of air condition system of airplane and distance of transect line datasets. More importantly, the distribution can provide modelling alternative for datasets that are skewed to the right.

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