DEVELOPMENT OF SINE-ODD FRÉCHET FAMILY OF DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

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ABSTRACT

This paper introduces a new family of continuous distributions, the Sine-Odd Fréchet-*G* family, which is derived by integrating the Odd Fréchet-*G* family into the Sine-*G* family of distribution. The novel distribution demonstrates significant flexibility, making it suitable for modeling datasets with different shapes and behaviors. The study examines the basic statistical properties of the distribution, and maximum likelihood estimation (MLE) is applied to estimate the model parameters. Two real-life datasets were analyzed to demonstrate the practical applicability of the distribution. The results of the analysis have shown that the Sine-Odd Fréchet-Weibull distribution offered a dominant fit compared to competing models. The distribution possessed the least values of Log-Likelihood, Akaike Information Criteria, Bayesian Information Criteria, Kolmogorov-Smirnov and highest p-value compared to the competitive models. These findings confirmed that the proposed distribution provides a more efficient and accurate fit to the datasets, in statistical modeling of lifetime and reliability data.

Keywords: MLE, Simulation, Bias, RMSE, Parameter and Distribution

1. INTRODUCTION

Statisticians have been developing new families of univariate distributions through the methods of Alzaatreh *et al.*, (2013). As a results of this effort, new distributions are obtained and developed by extending these new families of distributions with some common distributions to get more flexibility in modeling real life data (Sanusi *et al*., 2020). Probability distributions have been applied widely in different areas of application to model different types of datasets. Such areas are lifetime analysis, reliability, finance, economics, biological science, extreme events, medicine, agriculture, engineering and biological sciences among others (Almarashi *et al*., 2020; Nasiru *et al*., 2019). Many problems that abound in various field of human endeavour cannot be perfectly and adequately handled by most commonly known conventional probability distributions such as Normal, Weibull, Pareto, Gomepertz, Rayleigh, Exponential, e.t.c, (Yahaya and Doguwa (2021)).

Different probability distributions have been used widely to describe and forecast current occurrences in a variety of fields. However, the data in many of these areas typically exhibit complex behavior and a variety of forms that are linked to varying levels of skewness and kurtosis. For these reasons there is a need to extend the classical distributions to enhance their capability and at the same time improve their goodness of fit. Majority of the strategies are focused toward constructing heavy-tailed distributions, monotonic and non- monotonic failure rates, a tractable cumulative distribution function for easy simulation, and modelling data with varying degrees of skewness and kurtosis (Adekunle *et al 2024*).

Recently, some authors made attempts to define new trigonometric probability distributions to extend some popular distributions with greater flexibility in modeling different datasets. Some of the recently popular developed trigonometric families of distributions include, Exponentiated Sine-*G* by Muhammad *et al.*, (2021), Arctan X family of distribution by Alkhairy *et al.*, (2021), The Tan-*G* class by Souza *et al.*, (2021), Arcsine Exponentiated-*G* by He *et* al., (2020), Sine Inverse Lomax-*G* family by Famoye *et al.*, (2021), Sine Topp Leone-*G* by Al Babtain (2020), Tangent Topp-Leone *G* Family by Nanga *et al.*, (2022), Sine-*G* family by Sakthivel and Rajkumar (2021), Sine Alpha Power-*G* Family by Benchiha *et al.*, (2023). These distributions mentioned, are applied in describing real world phenomena. The theory of these distributions is widely studied which led to the development of new distributions. In this study therefore, a trigonometric sine transformation will be adopted to develop a new flexible family of distribution without the addition of other parameters.

2. SINE-ODD FRÉCHET (S-OFR)-*G* FAMILY OF DISTRIBUTION

Let X be a continuous random variable with a cumulative distribution function (CDF), F(x) and probability density function (PDF), *f*(x). The CDF of Sine- *G* family of distribution, according to Kumar *et al.*, (2015) is given as:

$$F(x,\xi)=\int_0^{\frac{\pi}{2}Q(x,\xi)}\cos t\,dt \;=\; sin\left[\frac{\pi}{2}Q(x,\xi)\right],\ \text{x} > 0 \tag{1}$$

$Q(x,\xi)$ is the CDF of the baseline distribution with vector ξ. Also, the CDF of the Odd Fréchet-*G* family of distributions according to Ulhaq and Elgarhy, (2018) is given as:

$$F_{OF-G}(x,\lambda,\xi)=\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right],\ x,\lambda>0 \tag{2}$$

The CDF of the Sine-Odd Fréchet (S-OFr) -*G* family of distributions is derived by applying the Sine transformation to the CDF of the Odd Fréchet-*G* distribution, and is given as:

$$F_{S-OFr-G}(x,\lambda,\xi)=\sin\left\{\frac{\pi}{2}\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right\},\text{x},\ \lambda>0 \tag{3}$$

where λ is the shape parameter, $G(x,\xi)$ is the CDF of any baseline *G*(x) and ξ is the vector of parameters. The PDF, *f*(x) of the S-OFr family, is given as:

$$f_{S-OFr-G}(x,\lambda,\xi)=\frac{\pi}{2}\frac{\lambda g(x,\xi)(1-G(x,\xi))^{\lambda-1}\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]}{G(x,\xi)^{\lambda+1}} \times\cos\left\{\frac{\pi}{2}\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right\} \tag{4}$$

where $g(x,\xi)$ is the PDF of any baseline distribution and ξ is the vector of the parameter. The hazard rate function of the S-OFr-*G* family is obtained as follows:

$$h(x,\lambda,\xi)=\frac{\frac{\pi}{2}\frac{\lambda g(x,\xi)(1-G(x,\xi))^{\lambda-1}\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]}{G(x,\xi)^{\lambda+1}}\cos\left\{\frac{\pi}{2}\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right\}}{1-\sin\left\{\frac{\pi}{2}\exp\left[-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right]\right\}} \tag{5}$$

The reduced form of the PDF in equation (4) which is important in the derivation of the statistical properties of the S-OFr-*G* family of distributions is given as:

$$f(x,\lambda,\xi)=\eta m g(x,\xi)G(x,\xi)^{m-1} \tag{6}$$

where $$\eta=\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}\pi^{2i+1}}{(2i+1)!2^{2i+1}} \tag{7}$$

and $$m=-\lambda j+k \tag{8}$$

3. STATISTICAL PROPERTIES OF S-OFR FAMILY OF DISTRIBUTIONS

3.1 Quantile function

The quantile function x_u of a random variable *X*, $0<u<1$ is defined as the inverse of the CDF. By equating the CDF F(x) in Equation (3) to a uniform variable u ∈ (0,1), and solving for x, the quantile function of the S-OFr-*G* distribution is obtained as:

$$x_u=Q(u)=G^{-1}(x,\xi)\left\{1+\left[-\log\left(\frac{\sin^{-1}(u)}{\frac{\pi}{2}}\right)\right]^{\frac{1}{\lambda}}\right\} \tag{9}$$

where G^{-1} denotes the quantile function of the baseline distribution.

3.2 Moments

The *r*th moment of a random variable *X* that follows a particular family of the distribution is given as:

$$\mu_r^{|}=E(X^r)=\int_{-\infty}^{\infty}x^r f(x)du \tag{10}$$

For the S-OFr-*G* family of the distribution, the moment is given as:

$$\mu_r^{|}=E(X^r)=\int_0^{\infty}x^r f_{S-OFrG}(x)dx=\int_0^{\infty}x^r\eta m\, g(x,\xi)G(x,\xi)^{m-1}dx \tag{11}$$

3.3 Moment Generating Function (MGF)

The moment generating function of a random variable *X* that follows the S-OFr-*G* family of the distribution is given as:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_{S-OFr-G}(x)dx = \eta m \int_0^\infty e^{tx} g(x,\xi)G(x,\xi)^{m-1}dx \tag{12}$$

3.4 Entropy

The entropy of the S-OFr-*G* family of distribution is given as:

$$I_R(\theta) = \frac{1}{1-\theta}\log\int_{-\infty}^{\infty} f(u)^\theta du \tag{13}$$

$$I_R(\theta) = \frac{1}{1-\theta}\log\left(\int_0^\infty \left(\eta m\, g(x,\xi)G(x,\xi)^{m-1}\right)^\theta dx\right) \tag{14}$$

$\theta > 0$ and $\theta \neq 1$

3.5 Order Statistics

Given that $x_1, x_2, \ldots, x_n$ is a random sample from the S-OFr-*G* family of distribution and $x_{i:n}$ represents the *i*th order statistics. The density function of the rth order statistics is given as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\eta m g(x,\xi)G(x,\xi)^{m-1}\left[\eta G(x,\xi)^m\right]\left[1-\eta G(x,\xi)^m\right]^{n-i} \tag{15}$$

4. ESTIMATION OF PARAMETER

The parameters of the S-OFr-*G* family of distributions are estimated in this section using the maximum likelihood estimation method.

Given a random sample, $x_1, x_2, \ldots, x_n$, of size *n* with parameters λ and ξ from S-OFr-*G* family of distributions. Suppose that $\Omega = [\lambda, \xi]^T$ to be [m x 1] vector of the parameter. Taking the log-likelihood function Ω using equation (4) is given as:

$$l = \log(\Omega) = n\log\frac{\pi}{2} + n\log(\lambda) + \sum\log g(x,\xi) + (\lambda-1)\sum\log(1-G(x,\xi))$$

$$-\sum\left(\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^\lambda\right)\right) - (\lambda+1)\sum\log G(x,\xi) + \sum\log\cos\left[\frac{\pi}{2}\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^\lambda\right)\right] \tag{16}$$

The partial derivative of the log-likelihood function in equation (16) gives the components of the score function $U\Omega = \left[\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \xi}\right]^T$ and is given as follows:

$$\frac{\partial l(\Omega)}{\partial \lambda}=\frac{n}{\lambda}+\sum \log(1-G(x,\xi))+\sum\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)\log\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)$$
$$-\sum \log G(x,\xi)+\frac{\pi}{2}\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\log\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)$$
$$\times\tan\left[\frac{\pi}{2}\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)\right] \tag{17}$$

$$\frac{\partial l(\Omega)}{\partial \xi}=\sum\frac{g'(x,\xi)}{G(x,\xi)}-(\lambda-1)\sum\frac{g(x,\xi)}{1-G(x,\xi)}+\sum\frac{\lambda g(x,\xi)}{G(x,\xi)^{2}}\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda-1}$$
$$-(\lambda+1)\sum\frac{g(x,\xi)}{G(x,\xi)}-\sum\frac{\pi}{2}\frac{\lambda g(x,\xi)}{G(x,\xi)^{2}}\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda-1}\exp\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)$$
$$\times\tan\left[\frac{\pi}{2}\left(-\left(\frac{1-G(x,\xi)}{G(x,\xi)}\right)^{\lambda}\right)\right] \tag{18}$$

The maximum likelihood estimation of the unknown parameters λ,ξ i.e., $\hat{\lambda},\hat{\xi}$ are the simultaneous solution of Equations (17) and (18), when the Equations are equated to zero as: $\frac{\partial\ell}{\partial\lambda}=0$ and $\frac{\partial\ell}{\partial\xi}=0$ and solved using iterative methods.

5. SUB-MODELS OF S-OFR *G* FAMILY OF DISTRIBUTION

In this section, a special case of the S-OFr-*G* family of distribution was developed in which Weibull was applied as a baseline model. The flexibility of the distributions in terms of their shapes for modelling different types of datasets is explored through the density and hazard rate plots for some selected parameters.

Given that the baseline distribution *G* has Weibull distribution with PDF and CDF, given in equations (19) and (20) respectively as:

$$g(x,\theta,k)=\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\qquad x,\theta,k>0 \tag{19}$$

$$G(x,\theta,k)=1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\qquad x,\theta,k>0 \tag{20}$$

This study now proposed S-OFr-Weibull distribution with PDF and CDF given in equations (21) and (22) and the plot of the two functions are given in Figure 1 and 2, respectively.

$$f(x,\lambda,k,\theta)=\frac{\pi}{2}\frac{\lambda\frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\left(1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)\right)^{\lambda-1}}{1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)^{\lambda+1}}$$

$$\times\exp\left[-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)}\right)^{\lambda}\right]\cos\left\{\frac{\pi}{2}\exp\left[-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)}\right)^{\lambda}\right]\right\} \tag{21}$$

$$x,\ \lambda,\ \theta,\ k>0$$

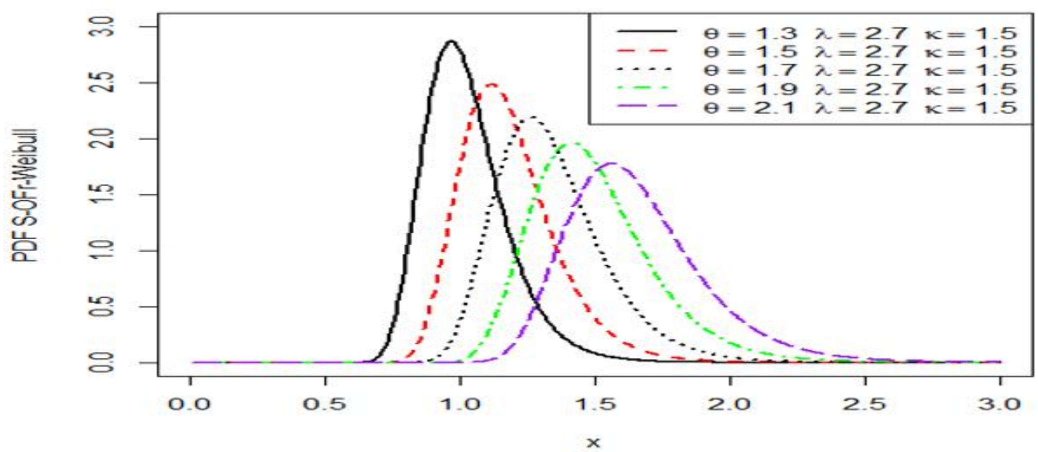

Figure 1: PDF Plot of Sine-Odd Fréchet-Weibull Distribution

$$F(x,\lambda,k,\theta)=\sin\left\{\frac{\pi}{2}\exp\left[-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)}\right)^{\lambda}\right]\right\} \tag{22}$$

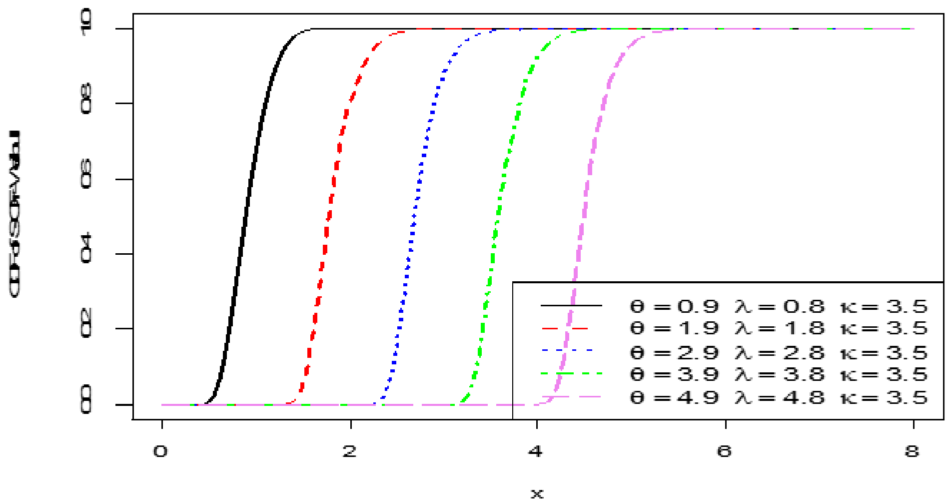

Figure. 2: CDF Plot of Sine Odd Fréchet-Weibull Distribution

The survival function, hazard function and quantile function of S-OFr-Weibull are given in Equation (23), (24) and (25) respectively.

$$S(x)=1-\sin\left\{\frac{\pi}{2}\exp\left[-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)}\right)^{\lambda}\right]\right\} \tag{23}$$

$$h(x)=\frac{\pi}{2}\frac{\lambda\left(\frac{x}{\theta}\right)^{k-1}\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\left(1-1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)^{\lambda-1}}{\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)^{\lambda+1}}\exp\left[-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)}\right)^{\lambda}\right]\cos\left\{\frac{\pi}{2}\exp\left[-\left(\frac{1-\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)\right)}{1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)}\right)^{\lambda}\right]\right\}$$
$$\times\left\{1-\sin\left[\frac{\pi}{2}\exp\left[-\left(\frac{1-1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)}{1-\exp\left(-\left(\frac{x}{\theta}\right)^{k}\right)}\right)^{\lambda}\right]\right]\right\}^{-1} \tag{24}$$

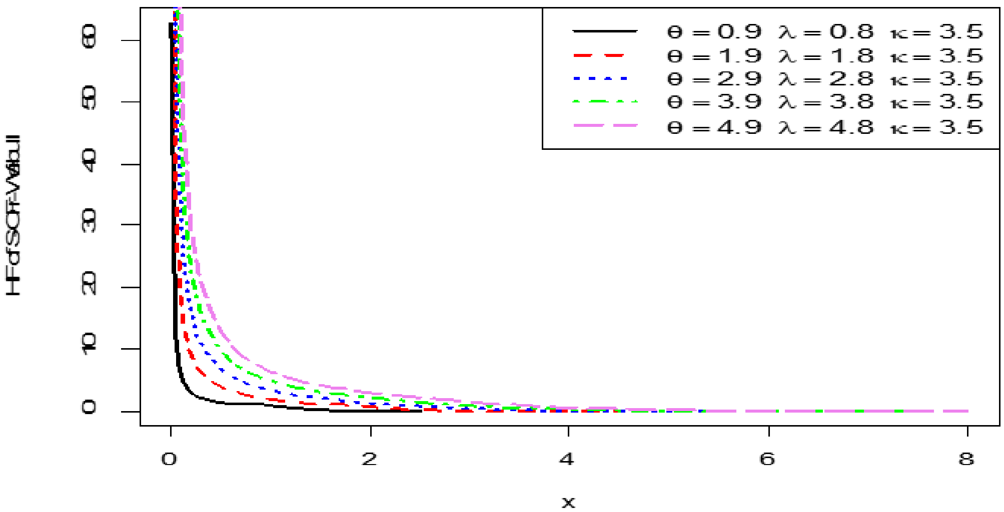

Figure 3: Hazard Function Plot of Sine-Odd Fréchet-Weibull Distribution

Figure 3 displays the plot of hazard function of S-OFr-Weibull distribution. The plot consists of monotonically decreasing curves, meaning the function starts at a high value and decreases as x increases. The curves have a steep decline initially, followed by a gradual leveling off. The shape suggests a heavy-tailed distribution, meaning that failures or events happen frequently at the beginning but become rare as time progresses. The different curves represent how different parameter values affect the distribution. The quantile function in Equation (24), also known as the inverse CDF, is essential for simulation as it allows random sampling from the distribution and supports Monte Carlo techniques which ensures realistic data generation in various fields. Many simulation methods rely on generating random numbers from a uniform distribution U(0,1).

$$x = \theta\left\{-\log\left[1+\left[1-\left[\log\left(\frac{\sin^{-1}(u)}{\frac{\pi}{2}}\right)\right]^{\frac{1}{\lambda}}\right]^{-1}\right]\right\}^{\frac{1}{k}} \tag{25}$$

6. SIMULATION STUDY

A simulation study was conducted using the Monte Carlo Simulation method to compute the mean, bias and root mean square error (RMSE) of the estimated parameters using the maximum likelihood estimation method. The stimulated data was generated using the quantile function defined in equation (25) for different sample sizes n =20, 50, 75, 100, 150 and 250 replicated 1000 times. For each sample size, the parameter values are $\theta = 2.3,\ \lambda = 2.4,\ k = 2.5$. Table 1 presents the results of the simulation with estimates, bias and RMSE from the proposed distribution.

Table 1: Results of the Simulated Data from S-OFr-Weibull Distribution

n	Properties	θ=2.3	λ=2.4	k=2.5
20	Estimate	2.5199	2.6314	2.2945
	Bias	0.2199	0.2314	-0.2055
	RMSE	0.5076	0.5875	0.2274
50	Estimate	2.5052	2.5192	2.3055
	Bias	0.2052	0.1192	-0.1945
	RMSE	0.4468	0.4562	0.2078
75	Estimate	2.4912	2.4948	2.3067
	Bias	0.1912	0.0948	-0.1933
	RMSE	0.3936	0.3891	0.2027
100	Estimate	2.5119	2.4650	2.3112
	Bias	0.2119	0.0650	-0.1888
	RMSE	0.3975	0.3700	0.1970
150	Estimate	2.4937	2.4554	2.3110
	Bias	0.1937	0.0554	-0.1890
	RMSE	0.3568	0.3114	0.1949
250	Estimate	2.4903	2.4489	2.3120
	Bias	0.1903	0.0489	-0.1880
	RMSE	0.3439	0.3010	0.1931

Table 1, presents the results obtained from the Monte Carlo simulation study from S-OFr-Weibull distribution. The results indicated that the bias and RMSE decrease toward zero with an increase in sample size. However, the actual value of the parameters and the estimated values are almost the same at different sample sizes and iterative levels for the MLE technique. These results support the consistency of the parameter estimates.

6.1 Application with Real Datasets

In this section, two real-life datasets are applied to demonstrate the performance of Sine-Odd Fréchet-Weibull distribution. Comparisons were made with Odd Fréchet Inverse Exponential (OFIE), Exponential Generalized Fréchet (EGF), Topp Leone Kumaraswamy Exponential (TLKE) and Weighted Weibull Exponential (WWE) distributions for their log-likelihood (LL), Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Kolmogorov Smirnov (KS) statistic.

The first dataset consists of 61 observations on the strength of carbon fibers tested under tension at gauge length of 10 mm. The dataset was used by Bi and Gui (2017).
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 4.024, 4.027, 4.225, 4.395, 5.020

Table 2: MLE Estimates, Information Criteria and Goodness of Fit Test for the strength of carbon fibers

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\kappa}$	$\hat{\beta}$	$\hat{\sigma}$	LL	AIC	BIC	KS (*p*-value)
SOFrW	**-**	**1.3513**	**3.5741**	**2.2032**	**-**	**-**	**52.7287**	**111.657**	**120.9072**	**0.0765 (0.8677)**
OFIE	1.8635	-	3.9773	-	-	-	56.3872	116.774	121.9961	0.1004 (0.5697)
EGF	10.1	1.5019	-	-	4.1022	4.3113	53.0498	114.1	114.8135	0.0894 (0.7145)
TLKE	8.4312	1.3872	6.5315	-	0.9164	-	52.9929	113.986	122.4293	0.0844 (0.7777)
WWE	0.0845	3.1235	4.2497	-	-	-	60.2645	126.529	126.9501	0.1066 (0.4920)

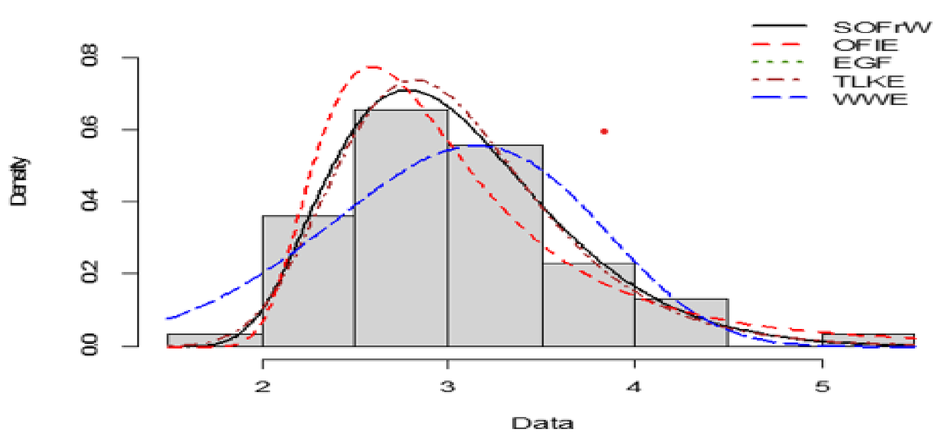

Figure 3: Density plot of Sine-Odd Frechet Weibull distribution

The second dataset consists of 54 observations of COVID-19 fatality ratio (%) of India, from 1*st* February to 1*st* April of year 2022. The data was collected from official site of World Health Organization (WHO)]. [https://covid19.who.int/
1.067, 1.757, 1.705, 1.849, 1.131, 1.595, 1.524, 1.014, 1.266, 1.678, 1.758, 1.898, 1.459, 3.370, 1.283, 1.753, 1.840, 2.134, 2.293, 2.217, 2.365, 1.485, 2.603, 2.952, 2.164, 3.142, 4.880, 2.885, 1.513, 2.704, 3.169, 2.485, 6.080, 2.462, 1.508, 1.078, 3.777, 3.407, 2.363, 5.893, 3.421, 7.211, 2.001, 2.087, 3.487, 3.457, 4.925, 2.469, 10.48, 2.514, 2.779, 2.514, 2.285, 3.895

Table **3:** MLE Estimates, Information Criteria and Goodness of Fit Test for COVID-19 fatality ratio (%) of India,

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\kappa}$	$\hat{\beta}$	$\hat{\sigma}$	LL	AIC	BIC	KS (*p*-value)
SOFrW		**3.6366**	**6.9934**	**0.343**			**82.1509**	**172.302**	**177.269**	**0.0480 (0.9996)**
OFIE	1.2687		1.7469				84.6486	173.532	178.275	0.0919 (0.7512)
EGF	1.8105	1.7872			0.8079	82.9668	83.9668	173.934	181.89	0.0572 (0.9940)
TLKE	7.4829	0.1893	2.2714		2.3251		85.0746	178.149	186.106	0.0829 (0.8513)
WWE	0.0523	5.1858	1.5544				96.4071	198.814	204.781	0.1715 (0.0834)

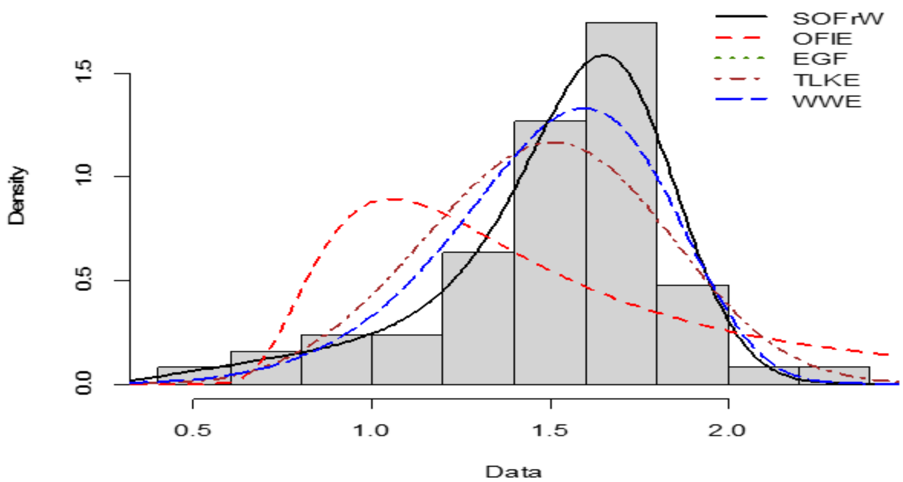

Figure 4: Density plot of Sine-Odd Frechet Weibull distribution

Tables 2 and 3 present the results of the analysis of the two datasets. The results of the analysis of Sine-Odd Fréchet Weibull were compared with Odd Fréchet Inverse Exponential Weibull (OFIE), Exponential Generalized Fréchet (EGF), Topp Leone Kumaraswamy Exponential (TLKE), and Weighted Weibull Exponential (WWE) distributions. The proposed Sine-Odd Fréchet-Weibull distribution has displayed good potential and has proven to be the better distribution because it has the least AIC, BIC and LL values. Also, the proposed model displays the least Kolmogorov Smirnov value with the highest corresponding p-value for the two datasets. To further validate the results obtained, the visual examination of the fit presented in Figures 4 and 5 also, confirms the superiority of the proposed distribution amongst its comparators. Thus, the proposed distribution fits the two datasets better.

7. CONCLUSION

This study introduced a Sine-Odd Fréchet family of distribution which is obtained by incorporating Odd Fréchet-*G* family into Sine-*G* family of distribution. This research explored the critical role of flexible lifetime models in effectively capturing the dynamic characteristics of real-world datasets. The model is characterized by high flexibility. The application of the two real-life datasets reveals that the model is strong as it fits better than its competitors. The resulting model is distinguished by its exceptional flexibility, enabling it to model a wide range of data behaviors with high precision. Based on the findings of this study, the development of flexible lifetime models has proven essential for more accurately capturing the characteristics of diverse datasets, enhancing the applicability and effectiveness of probability distributions. These findings also affirm the model's practical relevance and adaptability to complex data structures. This study hope that the proposed model will attract wider applications in different areas of research, and expect that the study can serve as a reference in advancing future research in this subject area.

REFERENCES

Adekunle, K. I., Abubakar, Y., Doguwa, S. I., & Yakubu, A. (2024). Kumaraswamy Type II Generalized Topp-Leone G Family of Distributions with Applications. *FUDMA Journal of Sciences*, 8(6), 186-195.

Al-Babtain, A. A., Elbatal, I., Chesneau, C., & Elgarhy, M. (2020). Sine Topp-Leone-G family of distributions: Theory and applications. *Open Physics*, 18(1), 574-593.

Alkhairy, I., Nagy, M., Muse, A. H. & Hussam, E. (2021). The Arctan-X family of distributions: Properties, Simulation, and Applications to Actuarial Sciences. *Complexity*, 1-14

Almarashi, A. M., Elgarhy, M., Jamal, F. & Chesneau, C. (2020). The exponentiated truncated inverse weibull-generated family of distributions with applications. *Symmetry*,12(4), 1-21

Alzaatreh A, Lee C, Famoye F. (2013) A New Method for Generating Families of distributions. *Metron* 71: 63- 79.

Benchiha, S., Sapkota, L. P., Al Mutairi, A., Kumar, V., Khashab, R. H., Gemeay, A. M., & Nassr, S. G. (2023). A New Sine Family of Generalized Distributions: Statistical Inference with Applications. *Mathematical and Computational Applications*, 28(4), 1-19.

Famoye, A., Algarni, A. & Almarashi, A. M. (2021). Sine Inverse Lomax Generated Family of Distributions with Applications. *Mathematical Problems in Engineering*, 1-11.

He, W., Ahmad, Z., Afify, A. Z. & Goual, H. (2020). The arcsine exponentiated-X family: validation and insurance application. *Complexity*, 1-18.

Kumar, D., Singh, U., & Singh, S. K. (2015). A new distribution using sine function-its application to bladder cancer patients' data. *Journal of Statistics Applications & Probability*, 4(3), 417-421

Muhammad, M., Alshanbari, H. M., Alanzi, A. R., Liu, L., Sami, W., Chesneau, C., & Jamal, F. (2021). A new generator of probability models: the exponentiated sine-G family for lifetime studies. *Entropy*, 23(11), 1-30.

Nanga, S., Nasiru, S., & Dioggban, J. (2022). Tangent Topp-Leone Family of Distributions. *Scientific African*, 17, e01363

Nasiru, S., Mwita, P. N., & Ngesa, O. (2018). Exponentiated generalized power series Family of distributions. *Annals of Data Science*, 6(3), 463–489

Sakthivel, K. M., & Rajkumar, J. (2021). Transmuted sine-G family of distributions: theory and applications. *Statistics and Applications*, 20(2), 73-92.

Sanusi A. A., Doguwa S. I, . Audu I. and Baraya Y. M. (2020) . Topp Leone Exponential – G Family of Distributions: Properties and Application. *NIPES Journal of Science and Technology Research* 2(4) 83-96

Souza, L., O Júnior, W. R. D., Brito, C. C. R. D., Chesneau, C., Fernandes, R. L., & Ferreira, T. A. (2021). Tan-G class of trigonometric distributions and its applications. *Cubo (Temuco)*, 23(1), 1-20.

Ul Haq, M. A., & Elgarhy, M. (2018). The Odd Frèchet-G Family of Probability Distributions. *Journal of Statistics, Applications and Probability*, 7(1), 189-203

Yahaya A. & Doguwa S.I.S. (2021). On Theoretical Study of Rayleigh-Exponentiated Odd Generalized X Family of Distributions. *Transactions of the Nigerian Association of Mathematical Physics, 14*, 143-154